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On the Conley-Zehnder index and Sasaki-Einstein manifolds

(Conley-Zehnder 지표와 Sasaki-Einstein 다양체에 대하여)

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On the Conley-Zehnder index and Sasaki-Einstein manifolds

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Abstract

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In the second chapter, we prove a useful relation between the Conley-Zehnder indices of the Reeb vector flow action along periodic orbits in prequantization bundles and the orbifold Chern class of the base symplectic orbifolds motivated by the well-known case of manifolds. We also apply this method to primary examples.

In the third chapter, we survey interesting properties on Sasaki-Einstein geometry from the elementary definitions and theorems to well-known examples and simple obstructions.

Key words: Conley-Zehnder index, orbifold Chern class, Brieskorn polynomial, Sasaki-Einstein manifold
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Contents

Chapter 1

Introduction

As a doctoral thesis, this article consists of two parts. The second chapter is mainly the introduction of [16] with additional preliminaries that is published during the author's degree work at Seoul National University. As for [16], there is a well-known relation between the generalized Conley-Zehnder indices or Robbin-Salamon indices of the Reeb vector flow along periodic orbits in prequantization bundles and the first Chern classes of base symplectic manifolds. The consequence of chapter 2 is almost parallel as that of Lemma 3.3 of [28] under the orbifold setting. That is, a symplectic manifold should be replaced by a symplectic orbifold, and the orbifold (co)homology or the orbifold Chern class plays a role in place of its counterpart in the usual sense.

In Section 2.3, we prove the main theorem in the orbifold case by a similar method used in Lemma 3.3 of [28]. Though some ingredients in Lemma 3.3 of [28] works like the same way, if we switch the situation to the orbifold setting, there are lots of subtleties that need to be tackled carefully because of it being an orbifold in lieu of being just a manifold. First of all, Reeb orbits in an orbibundle over a symplectic orbifold don't have identical periods in contrast to the manifold case. Despite similarities with the manifold case, not only does the consequence of this article provide us larger scope of applications but also we can enjoy interesting features of orbifolds while dealing with orbibundles, classifying spaces and their examples and so forth.

The algebraic varieties of weighted projective spaces and their complete intersections are well-known examples as orbifolds with special properties.

A great number of mathematicians have been and will be researching them using a variety of mathematical tools. We show that our main theorem works for those spaces and compute the actual values regarding them defined in the previous section.

For the links Y of weighted homogeneous polynomials, the mean Euler characteristics can be succinctly expressed as

$$\chi_m = (-1)^{\delta(Y)} \frac{\chi(IQ)}{|\mu_P|},$$

where $Q = Y/S^1$, IQ is the inertia orbifold of Q and the sign depends on the dimension of Y. By the consequence of this article the right hand side is expressed purely in terms of orbifold notions whereas the left hand side contains symplectic homology data only.

Although the main theorem works most effectively for principal orbits, it is still quite useful for non-principal ones as well, and we'll see it in section 2.3.4.

Additionally, it was shown that

$$\mu_P = 2 \operatorname{lcm}\{a_j\} \left(\sum_{j=0}^{n} \frac{1}{a_j} - 1 \right), \tag{1.0.1}$$

in the case of the Brieskorn polynomial of $\sum_{j=0}^{n} z_j^{a_j} = 0$, in [18]. The reader should be warned that they used the standard complex structure of $\mathbb{C}^{n+1}$ where the Brieskorn manifold resides as an affine hypersurface while computing the indices. In this article, however, we can't use that method as it is because every structure involved here must be invariant under the actions in order to be orbifold objects.

Also since the author has been working mostly on Sasaki-Einstein geometry, the third chapter is devoted to a survey of this area. Elementary materials on Sasaki-Einstein geometry are covered in this chapter from the basic definitions on Sasakian and Einstein geometry to examples constructed by some theoretic physicists in early 2000's. Because of the link between Sasaki-Einstein manifolds and the type IIB string theory, a lot of physicists produced interesting papers from a mathematical viewpoint as well regarding

Sasaki-Einstein geometry.

Kähler-Einstein geometry is also briefly addressed in section 3.2 because Sasakian geometry is strongly interconnected with Kähler geometry. For example, even though it is well-known that the first del Pezzo surface doesn't admit Kähler-Einstein metric, there exists a Sasaki-Einstein manifold whose cone metric coincides with the Kähler structure of the first del Pezzo surface, which intrigued the author a lot.

If a Kähler manifold or a Sasakian manifold carries a toric action, it gets even more interesting because we can extract lots of information on the manifold by looking into the image of the momentum map which happens to be a polyhedral cone. This is also overviewed in section 3.3.2.

A lot of basic computations are included in this article even for cases that are known to be true, because many of these cannot be found in the literature. Some computational steps that the original authors omitted in their papers because they thought to be trivial are also verified in this article. Not only is this article a good summary of the Sasaki-Einstein geometry but it also plays a role of a reference because it contains these rudimentary but exclusive computations.

Chapter 2

The Conley-Zehnder indices of the Reeb flow action along S^1-fibers over certain orbifolds

2.1 The Conley-Zehnder index

The Conley-Zehnder index is a crucial concept in researching the symplectic (co)homology and other related topics. For example, Otto van Koert introduced the notion of mean Euler characteristic in [29] as

$$\chi_m(W) = \frac{1}{2}\left(\liminf_{N\to\infty}\frac{1}{N}\sum_{i=-N}^{N}(-1)^i\, b_i\,(W) + \limsup_{N\to\infty}\frac{1}{N}\sum_{i=-N}^{N}(-1)^i\, b_i\,(W)\right),$$

with a simply connected Liouville filling $(W, d\lambda)$, $b_i = \mathrm{rk}SH_i^{+,S^1}(W)$ and also provided a simple and useful formula

$$\chi_m\,(W) = \frac{\sum_{i=1}^{k}(-1)^{\mu\left(\Sigma_{T_i}\right)-\frac{1}{2}\dim\left(\Sigma_{T_i}/S^1\right)}\phi_{T_i;T_{i+1},\ldots,T_k}\chi^{S^1}\left(\Sigma_{T_i}\right)}{|\mu_P|}, \tag{2.1.1}$$

where $\phi_{T_i;T_{i+1},\ldots,T_k} = \#\left\{a\in\mathbb{N} : aT_i < T_k \text{ and } aT_i \notin T_j\mathbb{N} \text{ for } j = i+1,\cdots,k\right\}$, and μ_P is the Robbin-Salamon index of a Reeb vector flow action along an orbit [18].

The terms 'Conley-Zehnder index' and 'Maslov index', however, seem to be being used for varied forms of definitions involved in paths or loops of symplectic matrices, and sometimes be used even interchangeably. So, we stick with the definitions in [25] for them in the sequel.

2.1.1 The Maslov index

Let us follow the definition provided in [23] for the Maslov index. First, we need some preliminary facts to define it. Also denote by $\mathrm{Sp}\,(2n)$ the group of symplectic $2n \times 2n$ matrices.

Proposition 2.1.1. *The unitary group $U\,(n)$ is a maximal compact subgroup of $\mathrm{Sp}\,(2n)$ and the quotient $\mathrm{Sp}\,(2n)\,/U\,(n)$ is contractible.*

Proposition 2.1.2. *The fundamental group of $U\,(n)$ is isomorphic to the integers. The determinant map $\det : U\,(n) \to S^1$ induces an isomorphism of fundamental groups.*

Theorem 2.1.3. *There exists a unique functor μ, called the Maslov index, which assigns an integer $\mu\,(\Psi)$ to every loop*

$$\Psi : \mathbb{R}/\mathbb{Z} \to \mathrm{Sp}\,(2n)$$

of symplectic matrices and satisfies the following axioms:

> **(homotopy)** *Two loops in $\mathrm{Sp}\,(2n)$ are homotopic if and only if they have the same Maslov index.*
>
> **(product)** *For any two loops $\Psi_1, \Psi_2 : \mathbb{R}/\mathbb{Z} \to \mathrm{Sp}\,(2n)$ we have*
>
> $$\mu\,(\Psi_1\Psi_2) = (\Psi_1) + (\Psi_2)\,.$$
>
> *In particular, the constant loop $\Psi\,(t) \equiv \mathbb{1}$ has Maslov index 0.*
>
> **(direct sum)** *If $n = n' + n''$ identify $\mathrm{Sp}\,(2n') \oplus \mathrm{Sp}\,(2n'')$ in the obvious way a subgroup of $\mathrm{Sp}\,(2n)$. Then*
>
> $$\mu\,(\Psi' \oplus \Psi'') = \mu\,(\Psi') + \mu\,(\Psi'')\,.$$

(normalization) *The loop* $\Psi : \mathbb{R}/\mathbb{Z} \to U(1) \subset \mathrm{Sp}(2n)$ *defined by* $\Psi(t) = e^{2\pi i}$ *has Maslov index* 1.

Explicitly

$$\mu(\Psi) = \deg \rho \circ \Psi,$$

where

$$\rho(\Psi) = \det\left(X + \sqrt{-1}Y\right),$$

$$\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} = \left(\Psi\Psi^T\right)^{-1/2} \in \mathrm{Sp}(2n) \cap O(n).$$

Using the Maslov index, we can define the first Chern numbers of symplectic bundles over Riemann surfaces. In [23], it is explained as follows:

Given a compact oriented Riemann surface Σ without boundary, choose a splitting

$$\Sigma = \Sigma_1 \cup_C \Sigma_2$$

such that $\partial\Sigma_1 = \partial\Sigma_2 = C$. Orient C as the boundary Σ_1. Now let E be a symplectic vector bundle over Σ and choose symplectic trivializations

$$\Sigma_k \times \mathbb{R}^{2n} \to E : (q, \zeta) \mapsto \Phi_k(q)\,\zeta$$

of E over Σ_1 and Σ_2. The overlap map $\Psi : C \to \mathrm{Sp}(2n)$ is defined by

$$\Psi(q) = \Phi_1(q)^{-1}\,\Phi_2(q)$$

for $q \in C$. Then the first Chern number $c_1(E)$ is defined by the Maslov index $\mu(\Psi)$ of Ψ.

2.1.2 The Conley-Zehnder index

The Maslov index works only for loops in $\mathrm{Sp}(2n)$. However we sometimes need to assign index numbers to paths in $\mathrm{Sp}(2n)$ which don't make a complete rotation. The Conley-Zehnder index is an integer $\mu_{CZ}(\Psi)$ assigned for a path $\Psi : [0,1] \to \mathrm{Sp}(2n)$ such that $\Psi(0) = \mathbb{1}$ and $\det(\mathbb{1} - \Psi(1)) \neq 0$. Any such path admits an extension $\Psi : [0,2] \to \mathrm{Sp}(2n)$, unique up to homotopy, such that $\Psi(s)$ does not have 1 as an eigenvalue for $s \geqslant 1$ and $\Psi(2)$ is one of the matrices

$$W^+ = -\mathbb{1}$$

or

$$W^- = \mathrm{diag}\,(-2, -1, \cdots, -1, 1/2, -1, \cdots, -1)\,.$$

Then Conley-Zehnder index is defined as

$$\mu_{CZ}(\Psi) = \deg \rho^2 \circ \Psi,$$

and has the following properties.

(naturality) For any path $\Psi : [0,1] \to \mathrm{Sp}\,(2n)$, $\mu_{CZ}(\Psi) = \mu_{CZ}(\Phi\Psi\Phi^{-1})\,.$

(homotopy) The Conley-Zehnder index is constant on the components of $\mathrm{SP}\,(n)$, the space of paths $\Psi : [0,1] \to \mathrm{Sp}\,(2n)$, such that $\Psi(0) = \mathbb{1}$ and 1 is not an eigenvalue of $\Psi(1)$.

(product) If $n = n' + n''$, identify $\mathrm{Sp}\,(2n') \oplus \mathrm{Sp}\,(2n'')$ in the obvious way a subgroup of $\mathrm{Sp}\,(2n)$. Then

$$\mu_{CZ}(\Psi' \oplus \Psi'') = \mu_{CZ}(\Psi') + \mu_{CZ}(\Psi'')\,.$$

(loop) If $\Phi : [0,1] \to \mathrm{Sp}\,(2n)$ is a loop with $\Phi(0) = \Phi(1) = \mathbb{1}$ then

$$\mu_{CZ}(\Phi\Psi) = \mu_{CZ}(\Psi) + 2\mu(\Phi)\,.$$

(inverse) $\mu_{CZ}(\Psi^{-1}) = \mu_{CZ}\left(\Psi^T\right) = -\mu_{CZ}(\Psi)\,.$

2.1.3 The Robbin-Salamon index

The Conley-Zehnder index may not be defined on the path $\Psi(t) = e^{2\pi i t}$, $t \in [0,1]$ in $U(1)$ which is one full rotation because $\det(\mathbb{1} - \Psi(1)) = 0$. J. Robbin and D. Salamon define another Maslov type index that includes such paths as full rotations [24]. They didn't name this index after their names in that article, it is called the Robbin-Salamon index usually. Also, some people phrase it as a generalized Conley-Zehnder index because it coincides with the Conley-Zehnder index in the case of paths satisfying $\det(\mathbb{1} - \Psi(1)) \neq 0$.

Let $\Psi : [a, b] \to \mathrm{Sp}(2n)$ be a path of symplectic matrices. Give $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ the symplectic form $\bar{\Omega} = -\Omega_0 \times \Omega_0$, and let Δ be the diagonal subspace of $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$. Now, let's call $t \in [a, b]$ is a crossing if $\dim \mathrm{Gr}\Psi(t) \cap \Delta \geqslant 1$. We define the Robin-Salamon index μ_{RS} as

$$\mu_{RS}(\Psi) = \frac{1}{2}\mathrm{sign}Q(a)|_{\mathrm{Gr}\Psi(a)\cap\Delta} + \sum_{a<t<b} \mathrm{sign}Q(t)|_{\mathrm{Gr}\Psi(t)\cap\Delta} + \frac{1}{2}\mathrm{sign}Q(b)|_{\mathrm{Gr}\Psi(b)\cap\Delta}$$

where the summation runs over all crossings t. Here, the quadratic form $Q(t_0)$ is defined by

$$Q(t_0)(v) = \left.\frac{d}{dt}\right|_{t=0} \bar{\Omega}(v, w(t)),$$

where $\mathbb{R}^{2n} \times \mathbb{R}^{2n} = \mathrm{Gr}\Psi(t_0) \oplus W$ is a Lagrangian splitting, $w(t) \in W$, and $v + w(t) \in \mathrm{Gr}\Psi(t)$

The Robin-Salamon index has the following properties.

(homotopy) μ_{RS} is invariant under homotopies with fixed endpoints.

(catenation) For $a < c < b$, $\mu_{RS}(\Psi) = \mu_{RS}\left(\Psi|_{[a,c]}\right) + \mu_{RS}\left(\Psi|_{[c,b]}\right)$.

(product) If $n = n' + n''$, identify $\mathrm{Sp}(2n') \oplus \mathrm{Sp}(2n'')$ in the obvious way a subgroup of $\mathrm{Sp}(2n)$, then

$$\mu_{RS}(\Psi' \oplus \Psi'') = \mu_{RS}(\Psi') + \mu_{RS}(\Psi'').$$

Example 2.1.4. *As an exercise, let's compute the Robbin-Salamon index for the one full rotation $\Psi(t) = e^{2\pi i t}$, $t \in [0, 1]$ in $U(1)$. As in the definition,*

$$\begin{aligned} \Delta =& \mathrm{Gr}\mathbb{1} = \{(z, z) : z \in \mathbb{C}\}, \\ W =& \mathrm{Gr}(-\mathbb{1}) = \{(z, -z) : z \in \mathbb{C}\}, \end{aligned}$$

then $\mathbb{C} \times \mathbb{C} = \Delta \oplus W$ is a Lagrangian splitting. Clearly there is no crossing in $0 < t < 1$. At $t = 0$, the Lagrangian path is

$$\mathrm{Gr}\left(e^{it}\right) = \left\{\left(z, e^{it} z\right) : z \in \mathbb{C}\right\}.$$

For $(1,1) \in \mathrm{Gr}\mathbb{1}$,

$$\left(z, e^{it}z\right) = (1,1) + (w, -w) \Rightarrow z = \frac{2}{1+e^{it}},\ \ w(t) = \frac{1-e^{it}}{1+e^{it}}.$$

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} \bar{\Omega}\left((1,1), (w,-w)\right) =& \left.\frac{d}{dt}\right|_{t=0} - dx \wedge dy\left(1, w(t)\right) + dx \wedge dy\left(1, -w(t)\right)\\ =& -\frac{1}{i}\left.\frac{d}{dt}\right|_{t=0}\left(w(t) - \bar{w}(t)\right)\\ =& -\frac{1}{i}\left.\frac{d}{dt}\right|_{t=0}\left(\frac{1-e^{it}}{1+e^{it}} - \frac{1-e^{-it}}{1+e^{-it}}\right)\\ =& 1.\end{aligned}$$

For $(i,i) \in \mathrm{Gr}\mathbb{1}$,

$$\left(z, e^{it}z\right) = (i,i) + (w, -w) \Rightarrow z = \frac{2i}{1+e^{it}},\ \ w(t) = i\frac{1-e^{it}}{1+e^{it}}.$$

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} \bar{\Omega}\left((i,i), (w,-w)\right) =& \left.\frac{d}{dt}\right|_{t=0} - dx \wedge dy\left(i, w(t)\right) + dx \wedge dy\left(i, -w(t)\right)\\ =& \left.\frac{d}{dt}\right|_{t=0}\left(w(t) + \bar{w}(t)\right)\\ =& \left.\frac{d}{dt}\right|_{t=0}\left(i\frac{1-e^{it}}{1+e^{it}} - i\frac{1-e^{-it}}{1+e^{-it}}\right)\\ =& 1.\end{aligned}$$

Everything should be the same at $t = 1$. $\therefore\ \mu_{RS}\left(\Psi(t)\right) = 2$.

In summary, the Robin-Salamon index for a path of $U(1)$ is

$$\mu_{RS}\left(e^{\pi i t}|_{t\in[0,T]}\right) = \begin{cases} T & \text{if } T \in 2\mathbb{Z},\\ 2\lfloor T/2 \rfloor + 1 & \text{otherwise.}\end{cases}$$

2.2 Orbifolds

2.2.1 Basic definitions

An orbifold is a generalization of a manifold. There are several ways to define an orbifold, but basically they refer to a manifold with *mild singularities*. The easiest way to define an orbifold is by using local uniformizing systems. The following definition can be found in [1] or [6] together with basic properties of general orbifolds.

Definition 2.2.1. Let X be a topological space, and fix $n \geqslant 0$.

- An n-dimensional orbifold chart on X is given by a connected open subset $\tilde{U} \subset \mathbb{R}^n$, a finite group Γ of smooth automorphism of $\tilde{U}$, and a map $\phi : \tilde{U} \to X$ so that ϕ is Γ-invariant and induces a homeomorphism of $\tilde{U}/\Gamma$ onto an open subset $U \subset X$.
- An embedding $\lambda : \left(\tilde{U}, \Gamma, \phi\right) \hookrightarrow \left(\tilde{V}, H, \psi\right)$ between two such charts is a smooth embedding $\lambda : \tilde{U} \hookrightarrow \tilde{V}$ with $\psi\lambda = \phi$.
- An orbifold atlas on X is a family $\mathcal{U} = \left\{\left(\tilde{U}, \Gamma, \phi\right)\right\}$ of such charts, which cover X and are locally compatible: given any two charts $\left(\tilde{U}, \Gamma, \phi\right)$ for $U = \phi\left(\tilde{U}\right) \subset X$ and $\left(\tilde{V}, H, \psi\right)$ for $V \subset X$, and a point $x \in U \cap V$, there exists an open neighborhood $W \subset U \cap V$ of x and a chart $\left(\tilde{W}, K, \mu\right)$ for W such that there are embeddings : $\left(\tilde{W}, K, \mu\right) \hookrightarrow \left(\tilde{U}, \Gamma, \phi\right)$, $\left(\tilde{W}, K, \mu\right) \hookrightarrow \left(\tilde{V}, H, \psi\right)$.
- An atlas $\mathcal{U}$ is said to refine another atlas $\mathcal{V}$ if for every chart in $\mathcal{U}$ there exists an embedding into some chart of $\mathcal{V}$. Two orbifold atlases are said to be equivalent if they have a common refinement.

Definition 2.2.2. An orbifold $\mathcal{X}$ of dimension n is a paracompact Hausdorff space X equipped with an equivalence class $[\mathcal{U}]$ of n-dimensional orbifold atlases.

We usually denote an orbifold by

$$\mathcal{X} = (X, \mathcal{U}),$$

where X is the base space and $\mathcal{U}$ is its orbifold atlas.

Definition 2.2.3. Let $x \in X$, where $\mathcal{X} = (X, \mathcal{U})$ is an orbifold. If $\left(\tilde{U}, \Gamma, \psi\right)$ is any local chart around $x = \psi(y)$, we define the local group at x as

$$\Gamma_x = \{g \in \Gamma : gy = y\}.$$

This group is uniquely determined up to conjugacy in Γ.

Definition 2.2.4. For an orbifold $\mathcal{X} = (X, \mathcal{U})$, we define its singular set as

$$\Sigma(\mathcal{X}) = \{X \in X : \Gamma_x \neq 1\}.$$

Definition 2.2.5. An effective quotient orbifold $\mathcal{X} = (X, \mathcal{U})$ is an orbifold given as the quotient of a smooth, effective, almost free action of a compact Lie group G on a smooth manifold M.

In the special case when

1. $\tilde{U} \cong \mathbb{C}^n$
2. Γ is a finite subgroup of $GL(n, \mathbb{C})$
3. all the embeddings are holomorphic,

then we call such an orbifold $\mathcal{X}$ a complex orbifold. In this case, the base space X is a complex space with quotient singularities at worst [6, p.123].

There is a notion of bundles over orbifolds. Whenever we generalize manifold notions to orbifolds, we need to require them to be compatible with the uniformizing group actions and hence they become quite complicated. The following is the definition in [6].

Definition 2.2.6. A V-bundle or orbibundle over an orbifold $(X, \mathcal{U})$ consists of a fiber bundle $B_{\tilde{U}}$ over $\tilde{U}$ for each chart $\left(\tilde{U}_i, \Gamma_i, \phi_i\right) \in \mathcal{U}$ with Lie group G and fiber F a smooth G-manifold which is independent of $\tilde{U}_i$ together with a homomorphism $h_{\tilde{U}_i} : \Gamma_i \to G$ satisfying

(i) if b lies in the fiber over $\tilde{x}_i \in \tilde{U}_i$ then for each $\gamma \in \Gamma_i$, $bh_{\tilde{U}_i}(\gamma)$ lies in the fiber over $\gamma^{-1}\tilde{x}_i$,

(ii) if the map $\lambda_{ji} : \tilde{U}_i \to \tilde{U}_j$ is an injection, then there is a bundle map $\lambda_{ji}^* : B_{\tilde{U}_j}|_{\lambda_{ji}(\tilde{U}_i)} \to B_{\tilde{U}_i}$ satisfying the condition that if $\gamma \in \Gamma_i$, and $\gamma' \in \Gamma_j$ is the unique element such that $\lambda_{ji} \circ \gamma = \gamma' \circ \lambda_{ji}$, then $h_{\tilde{U}_i}(\gamma) \circ \lambda_{ji}^* = \lambda_{ji}^* \circ h_{\tilde{U}_j}(\gamma')$, and if $\lambda_{kj} : \tilde{U}_j \to \tilde{U}_k$ is another such injection then $(\lambda_{kj} \circ \lambda_{ji})^* = \lambda_{ji}^* \circ \lambda_{kj}^*$.

If the fiber F is a vector space of dimension r and G acts on F as linear transformations of F, then it is called a vector V-bundle of rank r. Similarly, if F is the Lie group G with its right action, then it is called a principal V-bundle.

2.2.2 Classifying spaces

In order to define the classifying space of an orbifold, we need the notion of groupoid which is a bit abstract to understand. In fact, an orbifold is also defined as a proper, effective, étale groupoid and this definition will be tackled in one of the following sections. Instead of chasing all the notions necessary for the definition of a classifying space, let us consider the description provided in [6]:

Define the orthonormal frame bundle $\tilde{U}$ by $LO\left(\tilde{U}\right) = \bigsqcup_i LO\left(\tilde{U}_i\right)$. It is a principal $O(n)$-bundle over $\tilde{U}$. Recall that the universal $O(n)$-bundle $EO(n) \to BO(n)$ whose total space $EO(n)$ is a contractible space on which $O(n)$ acts freely. So we can construct the associated $O(n)$-bundle $\tilde{p} : LO\left(\tilde{U}\right) \times_{O(n)} EO(n) \to \tilde{U}$. This is just the universal space $E\mathcal{G} = LO\left(\tilde{U}\right) \times_{O(n)} EO(n)$. The differentials of the injections and the uniformizing group elements induce an action on $LO\left(\tilde{U}\right)$. By taking the quotient of this action, we get a V-bundle $p : LO(X) \times_{O(n)} EO(n) \to X$. We have a commutative diagram

$$\begin{tikzcd} E\mathcal{G} \arrow[rr, "\pi"] \arrow[d, "\tilde{p}"'] & & B\mathcal{G} \arrow[d, "p"] \\ \tilde{U} = \mathcal{G}_0 \arrow[rr, "\phi"'] & & \mathcal{G}_0/\mathcal{G} = X. \end{tikzcd}$$

Note that the map p is not a fibration. A generic fiber of p, that is a fiber over a regular point $x \in X$ is $EO(n)$, a contractible space, whereas, a singular fiber is an Eilenberg-MacLane space $K(\Gamma, 1)$, where Γ is the local uniformizing group.

Definition 2.2.7. We call $B\mathcal{G}$ the classifying space of the orbifold $\mathcal{X}$, and denote it by $B\mathcal{X}$.

There are several kinds of orbifold (co)homology groups, but the following is one of the simplest, which makes use of classifying spaces.

Definition 2.2.8. We define the orbifold cohomology, homology, and homotopy groups by

$$H^i_{orb}(\mathcal{X}, \mathbb{Z}) = H^i(B\mathcal{X}, \mathbb{Z}), H_i^{orb}(\mathcal{X}, \mathbb{Z}) = H_i(B\mathcal{X}, \mathbb{Z}), \pi_1^{orb}(\mathcal{X}) = \pi_1(\mathcal{X}).$$

Using classifying spaces, the concept of orbibundles is more easily understood.

Theorem 2.2.9. *Let $(\mathcal{X}, \mathcal{U})$ be an orbifold. There is a one-to-one correspondence between isomorphism classes of orbibundles on $\mathcal{X}$ with group G and generic fiber F and isomorphism classes of bundles on $B\mathcal{X}$ with group G and fiber F.*

Theorem 2.2.10. *The isomorphism classes of S^1 V-bundle E over an orbifold $(\mathcal{X}, \mathcal{U})$ are in one-to-one correspondence with elements of $H^2_{orb}(\mathcal{X}, \mathbb{Z})$, and the bijection is given by the orbifold first Chern class $p^* c_1^{orb}$. Furthermore, the total space of E is a smooth manifold if the local homomorphisms $h_{\tilde{U}_i}$ of the local uniformizing groups Γ_i mapping into the group $U(1)$ of the bundle are monomorphisms for all local uniformizing charts $\left(\tilde{U}_i, \Gamma_i, \phi_i\right)$.*

There is an orbifold analogue of the long exact homotopy sequence.

Theorem 2.2.11. *Let G be a compact Lie group that acts locally freely on an orbifold $\mathcal{Y} = (Y, \mathcal{V})$ with quotient orbifold $\mathcal{X} = (X, \mathcal{U})$. Then the sequence*

$$\cdots \to \pi_{i-1}^{orb}(G) \to \pi_i^{orb}(\mathcal{Y}) \to \pi_i^{orb}(\mathcal{X}) \to \pi_{i-1}^{orb}(G) \to \cdots .$$

of homotopy groups is exact.

Example 2.2.12. *Suppose $S^1 = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ acts on $S^3 = \{(z_0, z_1) \in \mathbb{C}^2 : |z_0|^2 + |z_1|^2 = 1\}$ by*

$$\zeta \cdot (z_0, z_1) = (\zeta z_0, \zeta^m z_1), \qquad \zeta \in S^1,$$

where m is an integer bigger than 1. Then the quotient space X under this action turns out to be a weighted projective space $\mathbb{P}(1, m)$ and with a suitable orbifold structure it becomes an orbifold, which we denote by $\mathcal{X}$ as an orbifold. We'll look into weighted projective spaces in one of the following sections.

Here, $p : B\mathcal{X} \to X$ is a singular fibration over $\mathcal{X}$, whose singular fiber over the north pole is the Eilenberg-MacLane space $K(\mathbb{Z}_m, 1)$ and the generic fiber is the infinite sphere except over the north pole. $H_2^{orb}(\mathcal{X}, \mathbb{Q})$ equals to $H_2(X, \mathbb{Q}) \cong \mathbb{Q}$ because $\mathbb{Q}$ is a field (see Corollary 4.3.8, [6]). To understand the natural projection $p_* : H_2^{orb}(\mathcal{X}, \mathbb{Q}) \to H_2(X, \mathbb{Q})$, think of $B\mathcal{X}$ made by attaching the boundary of a disk with degree m along the boundary of a small puncture on the north pole of the sphere. Then we can easily see that p_* induces the division by m in $\mathbb{Q}$.

Getting the orbifold (co)homology groups in $\mathbb{Z}$-coefficient is a little tricky. Consider the long exact sequence of the orbifold homotopy groups for an orbibundle $\mathcal{P}$ over $\mathcal{X}$ with the fiber F ([6, Theorem 4.3.18]):

$$\cdots \to \pi_n^{orb}(\mathcal{P}) \to \pi_n^{orb}(\mathcal{X}) \to \pi_{n-1}^{orb}(F) \to \cdots . \tag{2.2.1}$$

Now that our fiber F is S^1, it is easy to see $\pi_1(B\mathcal{X}) = 0, \pi_2(B\mathcal{X}) = \mathbb{Z}$, and hence $H_1(B\mathcal{X}) = 0, H_2(B\mathcal{X}) = \mathbb{Z}$. Now, let U be an open subset in $B\mathcal{X}$ that is the upper disk containing the north pole. i.e. it is a singular fiber bundle over a disk whose generic fiber is contractible and singular fiber is

$K(\mathbb{Z}_m, 1)$. Let V be the lower disk. i.e. it is the trivial bundle over a disk with a contractible fiber, and hence a contractible space. Then $U \cap V$ is a bundle over S^1 with a contractible fiber, and hence homotopy equivalent to S^1. Therefore,

$$\begin{aligned} H_*(U) &= H_*(K(\mathbb{Z}_m, 1)), \\ H_*(V) &= H_*(\{pt\}), \\ H_*(U \cap V) &= H_*(S^1). \end{aligned}$$

From the Mayer-Vietoris sequence, we conclude that

$$H_q^{orb}(\mathcal{X}) = \begin{cases} \mathbb{Z} & q = 0, 2 \\ \mathbb{Z}_m & q > 1 \ odd \\ 0 & q = 1 \ or \ q > 2 \ even \end{cases}.$$

Similarly, we can get the cohomology groups:

$$H^q_{orb}(\mathcal{X}) = \begin{cases} \mathbb{Z} & q = 0, 2 \\ \mathbb{Z}_m & q > 2 \ even \\ 0 & q > 0 \ odd \end{cases}.$$

2.3 The main theorem

2.3.1 The Boothby-Wang fibration

It is well-known fact that the set of principal circle bundles over a manifold M has a group structure isomorphic to the cohomology group $H^2(M, \mathbb{Z})$. Suppose that a symplectic manifold (M, Ω) has an integral symplectic form, i.e. $[\Omega] \in H^2(M, \mathbb{Z})$ and $\pi : P \to M$ is the corresponding S^1-bundle. Then there exists a connection form η on P such that $d\eta = \pi^*\omega$. We call such type of bundles as *Boothby-Wang fibrations* or *prequantization bundles.* Since the base space is a smooth manifold, the S^1-action should be free.

There is an orbifold analogue of this fact. The following theorem can be found in chapter 7 of [6].

Theorem 2.3.1. *Let $(\mathcal{Z}, \omega, J)$ be an almost Kähler orbifold with $[p^*\omega] \in$*

$H^2_{orb}(\mathcal{Z}, \mathbb{Z})$, and let M denote the total space of the circle V-bundle defined by the class $[\omega]$. Then the orbifold M admits a K-contact structure (ξ, η, Φ, g) such that $d\eta = \pi^*\omega$ where $\pi : M \longrightarrow \mathcal{Z}$ is the natural orbifold projection map. Furthermore, if all the local uniformizing groups of $\mathcal{Z}$ inject into the structure group S^1, then M is a smooth K-contact manifold.

Because the base space is an orbifold, the S^1-action is only locally free. i.e. all orbits of the S^1-action don't have to share a common period. So, for a prequantization bundle over an orbifold, we call a periodic orbit γ_P is *principal* if γ_P has the longest period among all the periodic orbits.

2.3.2 The main theorem

The main theorem will be proved in this section. According to Theorem 7.1.6 in [6], an integral almost Kähler orbifold admits a circle orbibundle generated by its symplectic form, whose total space becomes a K-contact orbifold. With counting the action by the Reeb vector flow along a fiber as a path of symplectomorphisms, we will compute its Maslov type index. The proof will be an orbifold version analogous to [28] with additional considerations.

Theorem 2.3.2. *Let $(\mathcal{Z}, \omega)$ be a Hodge orbifold with primitive $[p^*\omega]$, so that it admits an S^1-orbibundle $\pi : M \to \mathcal{Z}$ whose total space M has a K-contact structure $(\mathcal{R}_\eta, \eta, \Phi, g)$ where $d\eta = \pi^*\omega$. Further, if*

(i) $c_1^{orb}(T\mathcal{Z}) = -b_{\mathcal{Z}}[\omega] \in H^2(\mathcal{Z}, \mathbb{Q})$ *for some integer* $b_{\mathcal{Z}} \in \mathbb{Z}$

(ii) $\pi_1^{orb}(\mathcal{Z}) = 0$

(iii) *M is a manifold,*

then the Robbin-Salamon index μ_{RS} of the Reeb vector flow action along an orbit γ wound $|\Gamma_q|$-times is

$$\mu_{RS}(|\Gamma_q| \cdot \gamma) = 2b_{\mathcal{Z}},$$

where q is the image of γ under π and Γ_q is the isotropy group at q. For a principal orbit, we know that $|\Gamma_q| = 1$ and denote by $\mu_p(\mathcal{Z})$ its Robbin-Salamon index.

Proof. Before proceeding to prove the theorem, let us clarify what the path ϕ of symplectic matrices is whose index we look for. For a point x in $\mathcal{Z}$, let γ_x be the S^1-fiber on x or the periodic Reeb orbit, and for each point $t \in \gamma_x$, let Q^x_t be the horizontal space at t. First, fix any point in γ_x as 0. Since $[p^*\omega]$ is primitive and $\pi_1^{orb}(\mathcal{Z}) = 0$, M is simply connected because of the exact sequence of homotopy groups and hence there is a trivialization $\Phi(t) : \mathbb{C}^n \times \gamma(t) \to Q^x_t$ along $\gamma(t)$. Now let $\alpha^x(t) : Q^x_0 \longrightarrow Q^x_t$ be a symplectomorphism induced by the Reeb flow action from 0. Then, $\phi(t)$ is defined to be $\Phi^{-1}(t) \circ \alpha^x(t)^{-1} \circ \Phi(t)$. If the orbit γ happens to be principal, then ϕ becomes a loop. By [6, Proposition 7.5.23], $c_1(\mathcal{F}_\xi) = \pi^* c_1^{orb}(Z) = -b_{\mathcal{Z}}\pi^*[\omega] = -b_{\mathcal{Z}}\pi^*[d\eta]_B$. Hence the contact structure $\mathcal{D}$ has $c_1(\mathcal{D}) = 0$ in $H^2(M;\mathbb{Q})$ by [6, Proposition 7.5.26]. In other words, the index is independent of the choice of the trivialization.

Write $B\mathcal{Z} \xrightarrow{p} \mathcal{Z}$ for the classifying space of $\mathcal{Z}$. Choose points $q \in \mathcal{Z}$, $\tilde{q} \in T\mathcal{Z}$ such that $p(\tilde{q}) = q$ and write the periodic Reeb orbit over q by γ_q. As shown in [28], the Robbin-Salamon index along γ_q can be computed by considering the Maslov index along γ_q of the contact structure $\mathcal{D}$ of η as a symplectic bundle.

Due to Theorem 4.3.11 in [6], we can construct a $U(1)$-bundle $\tilde{M} \xrightarrow{\tilde{\pi}} B\mathcal{Z}$ over $B\mathcal{Z}$ corresponding to M, as in the following diagram

$$\begin{tikzcd} \tilde{M} \arrow[r, "\tilde{p}"] \arrow[d, "\tilde{\pi}"'] & M \arrow[d, "\pi"] \\ B\mathcal{Z} \arrow[r, "p"'] & \mathcal{Z}. \end{tikzcd}$$

Also, consider the vector bundle $\tilde{T}\mathcal{Z}$ over $B\mathcal{Z}$ corresponding to the tangent bundle $T\mathcal{Z}$ over $\mathcal{Z}$, as in the diagram

$$\begin{tikzcd} \tilde{T}\mathcal{Z} \arrow[d] & T\mathcal{Z} \arrow[d] \\ B\mathcal{Z} \arrow[r, "p"'] & \mathcal{Z}. \end{tikzcd}$$

Then, $\tilde{M}, \tilde{T}\mathcal{Z}$ are generic bundles over $B\mathcal{Z}$ and moreover, $p^* c_1^{orb}(T\mathcal{Z}) \in$

$H^2_{orb}(\mathcal{Z}, \mathbb{Z})$ is the first Chern class of $\tilde{T}\mathcal{Z}$.

Due to the assumption that $\pi_1^{orb}(\mathcal{Z}) = 0$, we know that

$$\pi_2^{orb}(\mathcal{Z}) = H_2^{orb}(\mathcal{Z}, \mathbb{Z}) \qquad \text{and} \qquad H^2_{orb}(\mathcal{Z}, \mathbb{Z}) = H_2^{orb}(\mathcal{Z}, \mathbb{Z})^*$$

by the universal coefficient theorem. We may pick a sphere

$$\iota : S \to B\mathcal{Z},$$

in $H_2(B\mathcal{Z}, \mathbb{Z})$ which is

$$\langle [p^*\omega], \iota(S) \rangle = 1,$$

because the Betti part of $H_2(B\mathcal{Z}, \mathbb{Z})$ is finite dimensional [6, Proposition 7.2.3].

Consider the pull-back bundle

$$\begin{array}{rl} P = & \iota^*\tilde{M} \\ & \downarrow{\scriptstyle \tilde{\pi}} \\ & S \end{array}$$

over S. Note that $(\iota\tilde{p})^*\eta$ is a connection 1-form and in order to get its curvature, compute

$$\tilde{\pi}^*\iota^*p^*\omega = \iota^*\tilde{\pi}^*p^*\omega = \iota^*\tilde{p}^*\pi^*\omega = \iota^*\tilde{p}^*d\eta.$$

Therefore, the first Chern class of P is

$$c_1(P) = [\iota^*p^*\omega].$$

Now, define the complex line bundle L by

$$L = P \times_{S^1} \mathbb{C}$$

over S, which is isomorphic to $\mathcal{O}(1)$ because of

$$\langle c_1(P), S \rangle = \langle [\iota^*p^*\omega], S \rangle = \langle [p^*\omega], \iota(S) \rangle = 1.$$

Hence we can take a section $\sigma : S \longrightarrow L$ that vanishes only at one point $\tilde{q}$ with multiplicity 1 and then extend it to a continuous map

$$\bar{\sigma} : D^2 \longrightarrow P.$$

Since π maps $\mathcal{D}$ symplectomorphically to $T_q\mathcal{Z}$ on every point of $\pi^{-1}(q)$, $\tilde{p}^*\mathcal{D}$ is mapped symplectomorphically to $\tilde{T}_{\tilde{q}}\mathcal{Z}$ via $\tilde{\pi}$ on every point of $\tilde{\pi}^{-1}(\tilde{q})$. Therefore, the Maslov index along γ_q of $\mathcal{D}$ is the same as the Maslov index along $\tilde{\gamma}_{\tilde{q}} = \tilde{p}^{-1}(\gamma_q)$ of $\tilde{p}^*\mathcal{D}$, so that the technique introduced in [28] for manifolds is still effective.

To be more precise, by considering trivializations of $\iota^*\tilde{T}\mathcal{Z}$ over a splitting $S = S\backslash\{\tilde{q}\} \cup \{\tilde{q}\}$ and its overlap map, we will relate the Maslov index along $\tilde{\gamma}_{\tilde{q}}$ with $p^*c_1^{orb}(T\mathcal{Z})$ as done in §2.6 of [23]. Since $S\backslash\{\tilde{q}\}$ is contractible, we may choose trivializations $\Phi : S\backslash\{\tilde{q}\} \times \mathbb{C}^n \cong \tilde{T}\mathcal{Z}|_{S\backslash\{\tilde{q}\}}$, and $\Psi : S\backslash\{\tilde{q}\} \times S^1 \cong P|_{S\backslash\{\tilde{q}\}}$. Write $\Psi^{-1}(\bar{\sigma}(\tilde{x})) = \{\tilde{x}\} \times \{t_{\tilde{x}}\} \in S\backslash\{\tilde{q}\} \times S^1$, and identify $\Theta_{\tilde{x}} : \tilde{T}_{\tilde{x}}Z \cong Q_0^x$ through $d\pi \circ p$, for $\tilde{x} \in S\backslash\{\tilde{q}\}$, $x = p(\tilde{x})$. Then by covering all $\tilde{x} \in S\backslash\{\tilde{q}\}$,

$$(\tilde{p}^*)^{-1} \circ \alpha^x(t_{\tilde{x}}) \circ \Theta_{\tilde{x}} \circ \Phi$$

gives a trivialization of $\tilde{p}^*\mathcal{D}$ over $\bar{\sigma}(S\backslash\{\tilde{q}\})$.

For the part of $\tilde{T}_{\tilde{q}}Z$, the horizontal lift along γ_q will work. Then, the overlap map with this splitting and trivializations is exactly the path of symplectomorphisms we pursue. In fact, this description requires a bit more precise verifications, which can be found in the proof of Lemma 3.3 of [28].

In the end, by the nature of the Maslov index, we have

$$\begin{aligned}
\mu_{RS}(\tilde{\gamma}_{\tilde{q}}) =& 2\left\langle c_1\left(\iota^*\tilde{T}\mathcal{Z}\right), S\right\rangle \\
=& 2\left\langle \iota^*c_1\left(\tilde{T}\mathcal{Z}\right), S\right\rangle \\
=& 2\left\langle c_1\left(\tilde{T}\mathcal{Z}\right), \iota(S)\right\rangle \\
=& 2\left\langle p^*c_1^{orb}(T\mathcal{Z}), \iota(S)\right\rangle \\
=& 2b_{\mathcal{Z}} \cdot \langle [p^*\omega], \iota(S)\rangle \\
=& 2 \cdot b_{\mathcal{Z}}.
\end{aligned}$$

Now that the degree of $\tilde{p}$ is equal to $|\Gamma_q|$, the order of the isotropy group of q in the base space, we get $\mu_{RS}(\tilde{\gamma}_{\tilde{q}}) = \mu_{RS}(|\Gamma_q| \cdot \gamma_q)$, which lead us to the conclusion. □

Remark 2.3.3. *As for the condition (ii), recall (2.2.1) the exact sequence of orbifold homotopy groups. If M happens to be a simply connected manifold, we have*

$$\pi_1(M) = \pi_1^{orb}(M) = 0,$$

and hence it follows that

$$\pi_1^{orb}(\mathcal{Z}) = 0.$$

2.3.3 The weighted projective spaces and their complete intersections

We denote by $\mathbb{P}(\mathbf{w})$ the weighted projective space with weights

$$\mathbf{w} = (w_0, w_1, \cdots, w_n),$$

and we assume additionally

$$\gcd(w_0, w_1, \cdots, w_n) = 1.$$

Let us use the following notations to make it easy in dealing with weighted projective spaces:

$$|\mathbf{w}| = \sum_{j=0}^{n} w_j,$$

$$\|\mathbf{w}\| = \prod_{j=0}^{n} w_j,$$

$$d_j = \gcd(w_0, \cdots, \widehat{w_j}, \cdots, w_n),$$

$$e_j = \operatorname{lcm}\left(d_0, \cdots, \widehat{d_j}, \cdots, d_n\right),$$

$$a_{\mathbf{w}} = \operatorname{lcm}(d_0, \cdots, d_n),$$

$$\bar{\mathbf{w}} = \left(\frac{w_0}{e_0}, \cdots, \frac{w_n}{e_n}\right).$$

Note that $\mathbf{w} = \bar{\mathbf{w}}$ if and only if $d_j = 1$ for all j or if and only if $a_{\mathbf{w}} = 1$. We call $\mathbb{P}(\mathbf{w})$ is *well-formed* in those cases.

In fact, an algebraic variety $\mathbb{P}(\mathbf{w})$ can have different orbifold structures, but in this article, we only consider the structure as a quotient space where S^1 acts locally free on S^{2n+1}. This well-known S^1-action is the generalization of the one introduced in Section 2.2.2:

$$\zeta \cdot (z_0, z_1, \cdots, z_n) \mapsto (\zeta^{w_0} z_0, \zeta^{w_1} z_1, \cdots, \zeta^{w_n} z_n), \zeta \in S^1.$$

The orbifold chart to define this orbifold structure can be found in p.143 [19]. Roughly; For $j \in \{0, \cdots, n\}$, let U_j be the subset of $\mathbb{P}(\mathbf{w})$ such that

$$U_j := \{[z_0 : \cdots : z_n] \mid z_j \neq 0\},$$

and $\tilde{U}_j$ the set of points in $\mathbb{C}^{n+1} \backslash \{\mathbf{0}\}$ such that $y_j = 1$. Then, $\tilde{U}_j$ is the local uniformizing chart with

$$\varphi_j : (y_0, \cdots, 1_j, \cdots, y_n) \in \tilde{U}_j \mapsto [y_0 : \cdots : 1_j : \cdots : y_n] \in U_j.$$

Its uniformizing group Γ_j is

$$\mu_{w_j} \cong \mathbb{Z}_{w_j},$$

where $\zeta \in \Gamma_j$ acts on $\tilde{U}_j$ by

$$\zeta \cdot (y_0, \cdots, 1_j, \cdots, y_n) \mapsto (\zeta^{w_0} y_0, \cdots, 1_j, \cdots, \zeta^{w_n} y_n).$$

The embeddings between local uniformizing charts are:

$$\psi_{ij} : (y_0, \cdots, 1_i, \cdots, y_n) \mapsto \left(y_0 / y_j^{w_0/w_j}, \cdots, 1_j, \cdots, y_n / y_j^{w_n/w_j} \right),$$

where y_j^{1/w_j} is a w_j-th root of y_j. We denote $\mathbb{P}(\mathbf{w})$ with this orbifold structure by $\mathcal{P}(\mathbf{w})$.

Remark 2.3.4. *There is another orbifold structure in $\mathbb{P}(\mathbf{w})$: The finite set $G_{\mathbf{w}} = \mathbb{Z}_{w_0} \times \cdots \times \mathbb{Z}_{w_n}$ acts on $\mathbb{CP}^n$ to produce a developable complex orbifold $\mathbb{CP}^n / G_{\mathbf{w}}$. Refer to [3] for this orbifold structure.*

Actually, $\mathcal{P}(\mathbf{w})$ is a symplectic orbifold: Consider the contact form in $\mathbb{C}^{n+1}$

$$\eta_{\mathbf{w}} = \frac{(2\pi)^{-1}\eta_0}{\sum_{j=0}^{n} w_j \left((x_j)^2 + (y_j)^2\right)},$$

where

$$\eta_0 = \frac{i}{2}\sum_{j=0}^{n}\left(z_j d\bar{z}_j - \bar{z}_j dz_j\right) = \sum_{j=0}^{n}\left(x_j dy_j - y_j dx_j\right).$$

Its Reeb vector field is

$$\mathcal{R}_{\mathbf{w}} = 2\pi i \sum_{j=0}^{n} w_j \left(z_j \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j}\right) = 2\pi \sum_{j=0}^{n} w_j \left(x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j}\right),$$

whose integral curve is

$$\left(e^{2\pi w_0 i t} z_0, \cdots, e^{2\pi w_j i t} z_j, \cdots e^{2\pi w_n i t} z_n\right).$$

Since $\eta_{\mathbf{w}}$ is invariant under S^1-action that defines $\mathcal{P}(\mathbf{w})$, we can use $d\eta_{\mathbf{w}}$ as an orbifold symplectic form ω for it.

Now, we show $\mathcal{P}(\mathbf{w})$ fits for Theorem 2.3.2 and get relevant values.

Lemma 2.3.5. *The above symplectic form $[\omega]$ amounts to $-1/\|\mathbf{w}\|$ in $H^2(\mathbb{P}(\mathbf{w}), \mathbb{Q}) \cong \mathbb{Q}$.*

Proof. Consider the map

$$f_{\mathbf{w}} : [z_0; \cdots; z_n] \in \mathbb{P}^n \mapsto [z_0^{w_0}; \cdots; z_n^{w_n}] \in \mathbb{P}(\mathbf{w}),$$

whose degree is $\|\mathbf{w}\| / \gcd\{w_j\}$ in general [19, Remark 3.5]. By the following commutative diagram

$$\begin{tikzcd}
\mathbb{P}^2 \arrow[r, "\iota"] \arrow[d, "f_{(w_0,w_1)}"'] & \mathbb{P}^n \arrow[d, "f_{\mathbf{w}}"] \\
\mathbb{P}(w_0, w_1) \arrow[r, "\iota_w"'] & \mathbb{P}(\mathbf{w}),
\end{tikzcd}$$

where

$$\iota, \iota_w : [z_0; z_1] \mapsto [z_0; z_1; 0; \cdots; 0],$$

and $\deg \iota = 1$, $\deg f_{(w_0,w_1)} = w_0 w_1 / \gcd(w_0, w_1)$, we see that $\deg \iota_w = \frac{\|\mathbf{w}\| \cdot \gcd(w_0,w_1)}{w_0 w_1}$. Also,

$$\iota_w^* \eta_{\mathbf{w}} = -\frac{i}{4\pi} \frac{\bar{z}_0 dz_0 - z_0 d\bar{z}_0 + \bar{z}_1 dz_1 - z_1 d\bar{z}_1}{w_0 |z_0|^2 + w_1 |z_1|^2}.$$

Now, use the chart $z \mapsto [z; 1; 0 \cdots ; 0]$ in $\mathbb{P}(w_0, w_1)$, so that

$$\begin{aligned} \iota_w^* d\eta_{\mathbf{w}} &= -\frac{i}{4\pi} \frac{(2w_1 dz \wedge d\bar{z})}{(w_0 |z|^2 + w_1)^2} \\ &= -\frac{i}{2\pi} \frac{(-2iw_1 dx \wedge dy)}{(w_0 |z|^2 + w_1)^2} \\ &= -\frac{1}{\pi} \frac{(w_1 r dr \wedge d\theta)}{(w_0 r^2 + w_1)^2}. \end{aligned}$$

The integral is, then

$$-\frac{1}{\pi} \int_0^{2\pi} \int_0^{\infty} \frac{w_1 r}{(w_0 r^2 + w_1)^2} dr d\theta = -\frac{1}{2\pi} \int_0^{2\pi} \int_{w_1}^{\infty} \frac{w_1}{w_0 s^2} ds d\theta = -\frac{1}{w_0}.$$

Since the uniformizing group of this chart is of order $w_1 / \gcd(w_0, w_1)$, the integral value should be $-\frac{\gcd(w_0,w_1)}{w_0 w_1}$. By factoring in the degree of ι_w, we get

$$\langle \omega, S^2 \rangle = -\frac{1}{\|\mathbf{w}\|}.$$

□

Lemma 2.3.6. *The isomorphism*

$$p^* : H^2(\mathbb{P}(\mathbf{w}); \mathbb{Q}) \longrightarrow H^2(B\mathcal{P}(\mathbf{w}); \mathbb{Q})$$

induced by the classifying space map $p : B\mathcal{P}(\mathbf{w}) \to \mathcal{P}(\mathbf{w})$ is the multiplication by $\|\mathbf{w}\|$.

Proof. Let

$$\mathbf{w_1} = (1, w_0, w_1, \cdots, w_n)$$

be another weight vector. Consider the following diagram

$$\begin{tikzcd}
\mathbb{P}^n \arrow[r, "\iota"] \arrow[d, "f_{\mathbf{w}}"'] & \mathbb{P}^{n+1} \arrow[d, "f_{\mathbf{w_1}}"] \\
\mathbb{P}(\mathbf{w}) \arrow[r, "\iota_w"'] & \mathbb{P}(\mathbf{w_1})
\end{tikzcd}$$

where

$$\iota, \iota_w : [z_0 : z_1 : \cdots : z_n] \mapsto [0 : z_0 : z_1 : \cdots : z_n],$$

and $f_{\mathbf{w}}, f_{\mathbf{w_1}}$ are defined as in Lemma 4.2. Because of

$$\deg \iota = 1, \ \deg f_{\mathbf{w}} = \deg f_{\mathbf{w_1}} = \|\mathbf{w}\|$$

[19, Remark 3.5], it follows that

$$\deg \iota_w = 1.$$

Since ι is a good orbifold map [19, Proposition 3.3], there is a map $B\iota$ between the classifying spaces induced by ι [6, p.119]

$$\begin{tikzcd}
B\mathcal{P}(\mathbf{w}) \arrow[r, "B\iota"] \arrow[d, "p"'] & B\mathcal{P}(\mathbf{w_1}) \arrow[d, "p_1"] \\
\mathcal{P}(\mathbf{w}) \arrow[r, "\iota_w"'] & \mathcal{P}(\mathbf{w_1}).
\end{tikzcd}$$

Meanwhile, we see the following exact sequence

$$\cdots \to \pi_2^{orb}(S^{2n+1}) \to \pi_2^{orb}(\mathcal{P}(\mathbf{w})) \to \pi_1^{orb}\left(S^1\right) \to \pi_1^{orb}\left(S^{2n+1}\right) \to \cdots,$$

which means

$$\pi_2^{orb}(\mathcal{P}(\mathbf{w})) = \mathbb{Z}.$$

Also, due to $\pi_1^{orb}(\mathcal{P}(\mathbf{w})) = 0$, we have

$$H_2^{orb}(\mathcal{P}(\mathbf{w})) = H_2^{orb}(\mathcal{P}(\mathbf{w_1})) = \mathbb{Z},$$

and hence

$$H^2_{orb}\left(\mathcal{P}\left(\mathbf{w}\right)\right) = H^2_{orb}\left(\mathcal{P}\left(\mathbf{w_1}\right)\right) = \mathbb{Z}.$$

Let us regard the classifying space as a singular fibration [6, p.117]. Then, it is clear that $B\iota$ is an embedding in the usual sense, and that the map α in the following diagram must be the identity map.

$$\begin{tikzcd}
\pi_2^{orb}\left(\mathcal{P}\left(\mathbf{w}\right)\right) \arrow[r, "B\iota_*"] \arrow[d] & \pi_2^{orb}\left(\mathcal{P}\left(\mathbf{w_1}\right)\right) \arrow[d] \\
\pi_1\left(S^1\right) \arrow[r, "\alpha"'] & \pi_1\left(S^1\right)
\end{tikzcd}$$

Consequently,

$$B\iota^* : H^2_{orb}\left(\mathcal{P}\left(\mathbf{w}\right)\right) \longrightarrow H^2_{orb}\left(\mathcal{P}\left(\mathbf{w_1}\right)\right)$$

is the identity map. Therefore

$$\deg p_1 = \deg p,$$

which allows us to assume that $w_0 = 1$.

Now, take a look at the following diagrams

$$\begin{tikzcd}
\mathbb{P}^2 \arrow[r, "\iota"] \arrow[d, "f_{(w_0,w_1)}"'] & \mathbb{P}^n \arrow[d, "f_{\mathbf{w}}"] \\
\mathbb{P}\left(w_0, w_1\right) \arrow[r, "\iota_w"'] & \mathbb{P}\left(\mathbf{w}\right),
\end{tikzcd}$$

where

$$\iota, \iota_w : \left[z_0; z_1\right] \mapsto \left[z_0; z_1; 0; \cdots; 0\right],$$

and

$$\begin{tikzcd}
B\mathcal{P}\left(w_0, w_1\right) \arrow[r, "B\iota"] \arrow[d, "p_1"'] & B\mathcal{P}\left(\mathbf{w}\right) \arrow[d, "p"] \\
\mathcal{P}\left(w_0, w_1\right) \arrow[r, "\iota_w"'] & \mathcal{P}(\mathbf{w}).
\end{tikzcd}$$

In this case, we have

$$\deg \iota = 1,\ \deg f_{(w_0,w_1)} = w_1/1 = w_1,$$

and hence

$$\deg \iota_w = \|\mathbf{w}\| / w_1.$$

By the similar argument, we see that

$$B\iota^* : H^2_{orb}(\mathcal{P}(\mathbf{w})) \longrightarrow H^2_{orb}(\mathcal{P}(w_0, w_1))$$

is the identity map.

On the other hand, we can show that

$$\deg p_1 = \operatorname{lcm}(w_0, w_1) = w_1,$$

by direct computations. Therefore, we have

$$\deg p = \|\mathbf{w}\|.$$

□

In order to figure out the orbifold canonical line bundle $\mathcal{K}^{orb}_{\mathcal{O}_{\mathcal{P}(\mathbf{w})}}$, we need to consider some line orbibundles $\mathcal{O}_{\mathcal{P}(\mathbf{w})}$ over $\mathcal{P}(\mathbf{w})$. The sheaves $\mathcal{O}_{\mathbb{CP}(\mathbf{w})}(m)$ on $\mathbb{CP}(\mathbf{w})$ generated by the graded $S(\mathbf{w})$-modules $S(\mathbf{w})(m)$ induce the orbisheaves $\mathcal{O}_{\mathcal{P}(\mathbf{w})}(m)$, which happen to be free hence line orbibundles and satisfy the property

$$\mathcal{O}_{\mathcal{P}(\mathbf{w})}(k) \otimes \mathcal{O}_{\mathcal{P}(\mathbf{w})}(l) = \mathcal{O}_{\mathcal{P}(\mathbf{w})}(k+l),\ k, l \in \mathbb{Z}, \tag{2.3.1}$$

none of which are true for $\mathcal{O}_{\mathbb{CP}(\mathbf{w})}(m)$ (refer to [6, Theorem 4.5.4]) in general.

Proposition 2.3.7. $\mathcal{P}(\mathbf{w})$ *satisfies the conditions in Theorem 2.3.2. Also, we have*

$$\mu_P(\mathcal{P}(\mathbf{w})) = 2\ |\mathbf{w}|.$$

Proof. Since the line orbibundle generated by $[\omega]$ in $\mathcal{P}(\mathbf{w})$ is actually $\mathcal{O}_{\mathcal{P}(\mathbf{w})}(-1)$, it is sufficient to compare $\mathcal{O}_{\mathcal{P}(\mathbf{w})}(-1)$ with $\mathcal{K}^{orb}_{\mathcal{P}(\mathbf{w})}$. Plus, it is known that the canonical divisor $K_{\mathbb{P}(\mathbf{w})}$ of the base space, which induces the canonical orbidivisor $\mathcal{K}^{orb}_{\mathcal{P}(\mathbf{w})}$, is equal to $\mathcal{O}_{\mathbb{P}(\mathbf{w})}(-|\mathbf{w}|)$. Because of (2.3.1), it is clear that

$$c_1^{orb}(\mathcal{P}(\mathbf{w})) = -|\mathbf{w}| \times c_1^{orb}\big(\mathcal{O}_{\mathcal{P}(\mathbf{w})}(-1)\big) = -b_{\mathcal{P}(\mathbf{w})} c_1^{orb}\big(\mathcal{O}_{\mathcal{P}(\mathbf{w})}(-1)\big) = -b_{\mathcal{P}(\mathbf{w})}[\omega].$$

Therefore, $b_{\mathcal{P}(\mathbf{w})} = |\mathbf{w}|$. □

Any quasi-smooth weighted complete intersection X in $\mathbb{P}(\mathbf{w})$ has an orbifold structure naturally induced by $\mathcal{P}(\mathbf{w})$ ([6, Proposition 4.6.6]). Denote X with this orbifold structure by $\mathcal{X}$.

Also, it is known that the link

$$L_X = S^{2n+1} \cap CX$$

with dim $\geqslant 2$ is simply connected [9, 3.2.12]. Since the locally free action of S^1 on L_X produces the orbifold $\mathcal{X}$, and L_X is a manifold in this case, we see that

$$\pi_1^{orb}(\mathcal{X}) = 0,$$

by the exact sequence (2.2.1).

Clearly, $\mathcal{X}$ is a symplectic orbifold with the inherited symplectic form $\iota^*\omega$ from $\mathcal{P}(\mathbf{w})$.

Proposition 2.3.8. *Let $\mathcal{X}$ be a quasi-smooth weighted complete intersection of $\mathcal{P}(\mathbf{w})$, whose degree is*

$$(m_1, m_2, \cdots m_r)$$

with $1 \leqslant r \leqslant n-2$. Then, we have

$$\mu_P(\mathcal{X}) = 2\left(|\mathbf{w}| - \sum_{j=1}^{r} m_j\right).$$

Proof. Owing to the condition $r \leqslant n-2$, we have $\dim X \geqslant 2$, and hence $\pi_1^{orb}(\mathcal{X}) = 0$. Consider the injective map induced by the embedding

$$\iota^* : H^2(\mathbb{P}(\mathbf{w}), \mathbb{Q}) \cong \mathbb{Q} \longrightarrow H^2(X, \mathbb{Q})$$

[9, (B22)Theorem].

Note that the local uniformizing chart $\tilde{U}_j$ of $\mathcal{P}(\mathbf{w})$ is equal to

$$Spec\left(k\left[z_0, \cdots, \widehat{z_j}, \cdots, z_n\right]\right),$$

$(k = \mathbb{C})$, and that $U_j = Spec\left(k\left[z_0, \cdots, z_n\right]_{(z_j)}\right)$ is the quotient space $\tilde{U}_j/\mathbb{Z}_{w_j}$ ([3, Theorem 3A.1.]). Also $\mathcal{O}_{\mathcal{P}(\mathbf{w})}(m)|_{\tilde{U}_j}$ is generated by $\left(\sqrt[w_j]{z_j}\right)^m$ over $\mathcal{O}_{\tilde{U}_j}$ ([6, Theorem 4.5.4]), and $W_j = C_X \cap \tilde{U}_j$ is a smooth affine variety for the affine quasicone C_X of X ([11, §3.1]).

Consequently, we may use W_j as a local uniformizing chart for the orbifold $\mathcal{X}$, on which $\mathcal{O}_{\mathcal{X}}(m)|_{W_j}$ is equal to $\mathcal{O}_{\mathcal{X}}|_{W_j} \otimes_{\mathcal{O}_{\tilde{U}_j}} \mathcal{O}_{\mathcal{P}(\mathbf{w})}(m)|_{\tilde{U}_j}$, a sheaf of module generated by $\left(\sqrt[w_j]{z_j}\right)^m$ over $\mathcal{O}_{\mathcal{X}}|_{W_j}$. Therefore, $\mathcal{O}_{\mathcal{X}}(m)$ is an invertible orbisheaf and

$$\mathcal{O}_{\mathcal{X}}(m) \otimes \mathcal{O}_{\mathcal{X}}(l) = \mathcal{O}_{\mathcal{X}}(m + l)$$

still holds. Because it is now clear that the line bundle generated by $\left[\iota^*\omega\right]$ is equal to $\mathcal{O}_{\mathcal{X}}(-1)$, and also we know that $K_X = \mathcal{O}_X\left(\sum_{j=1}^r m_j - |\mathbf{w}|\right)$ ([11, 3.3.4 Theorem]), we may let $c_1^{orb}\left(-\mathcal{K}_{\mathcal{X}}^{orb}\right) = -b_{\mathcal{Z}}\left[\iota^*\omega\right] = -\iota^*\left(b_{\mathcal{Z}}\left[\omega\right]\right)$, so that

$$-b_{\mathcal{Z}}\left[\omega\right] = -b_{\mathcal{Z}} c_1^{orb}\left(\mathcal{O}_{\mathcal{P}(\mathbf{w})}\left(-1\right)\right) = c_1^{orb}\left(\mathcal{O}_{\mathcal{P}(\mathbf{w})}\left(|\mathbf{w}| - \sum_{j=1}^r m_j\right)\right),$$

which leads us to the conclusion. □

Now, turn our attention to the Brieskorn polynomial of the exponent vector

$$\mathbf{a} = \left(a_0, \cdots, a_n\right), n \geqslant 3.$$

i.e.

$$\sum_{j=0}^n z_j^{a_j} = 0.$$

Write $l = \mathrm{lcm}_j\{a_j\}$. Then, we may regard it as a quasi-smooth weighted complete intersection $\mathcal{X}$ of a single weighted homogeneous polynomial of degree l in the weighted projective space with weights $w_j = l/a_j$, and hence it has the orbifold structure induced by the standard weighted projective space (refer to [6, example 4.6.7]).

Here, let us define l_2 in the following way. First, let $l = p_1^{s_1} \cdots p_t^{s_t}$ be the prime factorization of l. For each prime p_j, choose a_{j_p} whose $\beta_j = \mathrm{ord}_p(a_{j_p})$ is the second largest among $\mathrm{ord}_p a_\alpha$'s. Put $l_2 = p_1^{\beta_1} \cdots p_t^{\beta_t}$.

Remark 2.3.9. *For* $\mathbf{a} = (a, a, \cdots, a)\,(n \geqslant 2)$ *for example, we have* $l = l_2 = a$. *If* $\mathbf{a} = (a_0, \cdots, a_n)$ *consists of pairwise relatively prime integers, then it is clear that*

$$l = a_0 \times \cdots \times a_n, \qquad l_2 = 1.$$

More concretely, for $\mathbf{a} = (p, p^2, p^3)$ *with* p *prime, we have* $l = p^3,\ l_2 = p^2$.

Lemma 2.3.10. *Under the above assumption, we have*

$$a_{\mathbf{w}} = l/l_2,$$

and hence $\mathcal{X}$ *is well-formed iff* $l = l_2$.

Proof. Note that each w_j divides l and so do all d_j's, which leads us to $a_{\mathbf{w}}|l$. In order to get $a_{\mathbf{w}}$, we need the biggest ord_p among d_j's, for each prime factor p of l. For each p_j, let α_j be the index where $\mathrm{ord}_{p_j}\left(a_{\alpha_j}\right) = s_j$. Then p_j does not divide w_{α_j}, and hence d_α is indivisible by p_j for all α different from α_j. Among w_α with $\alpha \neq \alpha_j$, $s_j - \beta_j$ is the least one of $\mathrm{ord}_p\left(w_\alpha\right)$. i.e.

$$\mathrm{ord}_p\left(d_{\alpha_j}\right) = s_j - \beta_j,$$

and hence

$$\mathrm{ord}_p\left(a_{\mathbf{w}}\right) = s_j - \beta_j.$$

Because of $l = l_2$, the ambient projective space is well-formed and so is its complete intersection of the Brieskorn hypersurface by the criterion introduced in Corollary 4.6.10 in [6]. □

Corollary 2.3.11. *Consider the Brieskorn orbifold with the exponent vector*

$$\mathbf{a} = \left(a_0, \cdots, a_n\right), n \geqslant 3$$

as a weighted hypersurface in the weighted projective space with weights $\{l/a_j\}$.

Then the Robin-Salamon index of the principal orbit is

$$\mu_P = 2\ l\left(\sum_{j=0}^{n}\frac{1}{a_j} - 1\right).$$

Proof. It follows directly from the preceding description. □

Remark 2.3.12. *This value matches (1.0.1) although their paths of symplectomorphisms are not defined exactly in the same manner.*

2.3.4 Some computations for non-principal orbits

In this section, we will see how Theorem 2.3.2 works for non-principal orbits as well by tackling the examples in the previous section.

Let us begin with the one-dimensional weighted projective space $\mathcal{P}(m,n)$ (m,n are relatively prime). Consider the non-principal orbit

$$\gamma(t) = \left(e^{2\pi i t}, 0\right), t \in [0,1],$$

in $\mathcal{P}(m,n)$. Clearly, it corresponds to the point x whose isotropy group is $\Gamma_x = \mathbb{Z}_m$. By Theorem 2.3.2, $\mu_{RS}(m\cdot\gamma) = 2(m+n)$, and it is obvious that

$$\mu_{CZ}(\gamma) = 2\left\lfloor\frac{m+n}{2m}\right\rfloor + 1,$$

because it is a path of one-dimensional unitary matrices.

Now, move on to the higher dimensional weighted projective space $\mathcal{P}(\mathbf{w})$ with weights $\mathbf{w} = (w_0, w_1, \cdots, w_n)$, and with Reeb vector field

$$\mathcal{R}_{\mathbf{w}} = 2\pi i\sum_{j=0}^{n} w_j\left(z_j\frac{\partial}{\partial z_j} - \bar{z}_j\frac{\partial}{\partial \bar{z}_j}\right).$$

Since $\mathcal{P}(\mathbf{w})$ has a compatible hermitian metric, the action by $\mathcal{R}_{\mathbf{w}}$ along a principal orbit can be represented as a loop of unitary matrices $A(t)$. Therefore, if we consider another vector field $\mathcal{R}_{\mathbf{w_0}} = 2\pi i\sum_{j=0}^{n-1} w_j\left(z_j\frac{\partial}{\partial z_j} - \bar{z}_j\frac{\partial}{\partial \bar{z}_j}\right)$

along the same orbit in $\mathcal{P}(\mathbf{w})$, then A can be written as

$$A(t) = \begin{bmatrix} A_0(t) & * \\ 0 & \varphi(t) \end{bmatrix},$$

where $A_0(t)$ is generated by $\mathcal{R}_{\mathbf{w_0}}$ and $\varphi(t)$ is a $\mathbb{C}$-valued function. Also, because $A(t), A_0(t)$ are unitary matrices,

$$1 = |\det A(t)| = |\det A_0(t)|\, |\varphi(t)| = |\varphi(t)|\,.$$

Consider the non-principal orbit $\gamma_0 = (e^{2\pi w_0 it}z_0, \cdots, e^{2\pi w_{n-1}it}z_{n-1}, 0)$ in $\mathcal{P}(\mathbf{w})$ with nonzero z_j's, which corresponds to the point whose isotropy group is $\mathbb{Z}_{d_n}$, where $d_n = \gcd(w_0, w_1, \cdots, w_{n-1})$. By applying Proposition 2.3.7 to $\mathcal{P}(\mathbf{w})$ and to one-dimensional lower weighted projective space $\mathcal{P}(\mathbf{w}_0)$ with weights $\mathbf{w_0} = (w_0, w_1, \cdots, w_{n-1})/d_n$, we know that $\varphi(t)$ contributes twice as much as $|\mathbf{w}| - d_n \cdot |\mathbf{w}_0| = w_n$ to the Robbin-Salamon index in $\mathcal{P}(\mathbf{w})$ along γ_0 traveling d_n-times repeatedly. Therefore, we get

$$\mu_{RS}(\gamma_0) = \frac{2}{d_n}\sum_{j=0}^{n-1} w_j + 2\left\lfloor \frac{w_n}{2d_n} \right\rfloor + 1.$$

By using this method repeatedly, we can get the Robbin-Salamon index for any non-principal orbits in the weighted projective spaces. For example, consider $\gamma_0 = (e^{2\pi it}, e^{2\pi it}, 0, 0)$, $t \in [0,1]$, in $\mathcal{P}(4,4,5,14)$. With an iteration of the above operation, we get

$$\mu_{RS}(\gamma_0) = 2\cdot(1+1) + (2\cdot\left\lfloor \frac{5}{2\cdot 4} \right\rfloor + 1) + (2\cdot\left\lfloor \frac{7}{2\cdot 2} \right\rfloor + 1) = 8.$$

after all.

It's even possible to apply this method for the Brieskorn cases. Consider $\gamma_0 = (e^{\pi it}z_0, e^{\pi it}z_1, e^{\pi it}z_2, 0)$, $t \in [0,1]$ in $z_0^2 + z_1^2 + z_2^2 + z_3^5 = 0$, $(z_0z_1z_2 \neq 0)$. Note that it is a quasi-smooth hypersurface with degree $d = \mathrm{lcm}(2,2,2,5) = 10$ in $\mathcal{P}(5,5,5,2)$, and γ_0 corresponds to a point with an isotropy group of order 5 when counting it with the induced orbifold structure. Therefore, if we make use of the similar method as above, we get

$$\mu_{RS}(\gamma_0) = 2 \cdot (1+1+1-2) + (2 \cdot \left\lfloor \frac{2}{2 \cdot 5} \right\rfloor + 1) = 3,$$

because of $\mu_P = 2 \cdot (5+5+5+2-10)$.

Remark 2.3.13. *We need to be careful in applying this method to non-principal orbits in general orbifolds, because we have implicitly used the fact that in $\mathbb{C}^n$, the subspaces $\{z_n = 0\}$ and $\{z_1 = \cdots = z_{n-1} = 0\}$ are unitarily complementary to each other so that they are not only orthogonal but also symplectic complement of each other. Apparently, this is not always true for general orbifold strata.*

2.3.5 Inertia orbifolds

We may rewrite (2.1.1) by using the inertia orbifold, whose main ingredient idea was provided by Otto van Koert. When defining inertia orbifolds, it is more convenient if we use the definition for orbifolds with the notion of groupoids. Let us consider definitions and examples in [1].

Definition 2.3.14. A topological groupoid $\mathcal{G}$ is a groupoid object in the category of topological spaces. That is, $\mathcal{G}$ consists of a space G_0 of objects and a space of G_1 of arrows, together with five continuous structure maps listed below.

1. The source map $s : G_1 \to G_0$, which assigns to each arrow $g \in G_1$ its source $s(g)$.

2. The target map $t : G_1 \to G_0$, which assigns to each arrow $g \in G_1$ its target $t(g)$. For two objects $x, y \in G_0$, one writes $g : x \to y$ or $x \xrightarrow{g} y$ to indicate that $g \in G_1$ is an arrow with $s(g) = x$ and $t(g) = y$.

3. The composition map $m : G_1 {}_s\times_t G_1 \to G_1$. If g and h are arrows with $s(h) = t(g)$, one can form their composition hg, with $s(hg) = s(g)$ and $t(hg) = t(h)$. If $g : x \to y$ and $h : y \to z$, then hg is defined and

$hg : x \to z$. The composition map, defined by $m(h,g) = hg$, is thus defined on the fibered product

$$G_1\,{}_s\!\times_t G_1 = \{(h,g) \in G_1 \times G_1 : s(h) = t(g)\},$$

and is required to be associative.

4. The unit map $u : G_0 \to G_1$, which is a two-sided unit for the composition. This means that $su(x) = x = tu(x)$, and that $gu(x) = g = u(y)\,g$ for all $x, y \in G_0$ and $g : x \to y$.

5. An inverse map $i : G_1 \to G_1$, written $i(g) = g^{-1}$. Here, if $g : x \to y$, then $g^{-1} : y \to x$ is a two-sided inverse for the composition, which means that $g^{-1}g = u(x)$ and $gg^{-1} = u(y)$.

Before defining the inertia orbifolds, we need to consider the following example:

Example 2.3.15. *Suppose a Lie group K acts smoothly on a manifold M from the left. One defines a Lie groupoid $K \ltimes M$ by setting $(K \ltimes M)_0 = M$ and $(K \ltimes M)_1 = K \ltimes M$, with $s : K \times M \to M$ the projection and $t : K \times M \to M$ the action. Composition is defined from the multiplication in the group K, in an obvious way. This groupoid is called the action groupoid or translation groupoid associated to the group action. The unit groupoid is the action groupoid for the action of the trivial group. On the other hand, by taking M to be a point we can view any Lie group K as a Lie groupoid having a single object.*

The inertia orbifold is a special case of the previous example:

Example 2.3.16. *Suppose that $\mathcal{G} = G \ltimes X$ is a global quotient groupoid. An important object is the so-called inertia groupoid $\wedge\mathcal{G} = G \ltimes (\sqcup_g X^g)$. Here X^g is the fixed point set of g, and G acts on $\sqcup_g X^g$ as $h : X^g \to X^{hgh^{-1}}$ given by $h(x) = hx$. The groupoid $\wedge\mathcal{G}$ admits a decomposition as a disjoint union: let $\wedge(\mathcal{G})_{(h)} = G \ltimes \left(\sqcup_{g\in(h)} X^g\right)$. If S is a set of conjugacy class representatives for G, then*

$$\wedge\mathcal{G} = \bigsqcup_{h\in S} (\wedge\mathcal{G})_{(h)}\,.$$

By our definition, the homomorphism $\phi : (\wedge\mathcal{G})_{(h)} \to \mathcal{G}$ induced by the inclusion maps $X^g \to X$ is an embedding. Hence, $\wedge\mathcal{G}$ and the homomorphism ϕ together form a (possibly non-disjoint) union of suborbifolds.

For the links of weighted homogeneous polynomials, (2.1.1) can be rewritten a bit more effectively. Let $G = S^1 = U(1)$ and X be the zero set in $\mathbb{C}^n$ of some Brieskorn polynomial. Write $\mathcal{G} = G \ltimes X$ for the orbifold as in Example 2.3.15 and $\mathcal{G}_j$ suborbifolds defined in Example 2.3.16. List $H_1, H_2, \cdots, H_k$ the finite cyclic subgroups of $U(1)$ with degrees m_i that are isotropy groups of some points, so that

$$m_1 \geqslant m_2 \geqslant \cdots \geqslant m_k = 1.$$

Of course, the denominator and $\mu(\Sigma_{T_i})$ are computed as in the previous section, and note that $H_i = \left\langle e^{2\pi\sqrt{-1}/m_i} \right\rangle$ and

$$\phi_{T_i;T_{i+1},\ldots,T_k} = \#\text{ of elements } H_i / (H_{i+1} \cup \cdots \cup H_{k-1}).$$

Therefore, the numerator can be rewritten as

$$\sum (-1)^{\mu\left(\Sigma_{T_i}\right) - \frac{1}{2}\dim(\mathcal{G}_i)} \chi^{S^1}(\Sigma_{T_i}),$$

where the sum runs over all suborbifolds $\mathcal{G}_i$ as in Example 2.3.16. In particular, there is an interesting theorem in [30].

Theorem 2.3.17. *Assume that p is a weighted homogeneous polynomial. Put $Y := L(p)$. This contact manifold is filled by a Stein subdomain of $W = p^{-1}(1)$. In the case that the quotient orbifold $Q = Y/S^1$ is Fano or of general type, the mean Euler characteristic can be computed as*

$$\chi_m(W) = (-1)^{\delta(Y)} \frac{\chi(IQ)}{|\mu_P|},$$

where IQ is the inertia orbifold of Q and

$$\delta(Y) = \begin{cases} 0, & \dim Y \equiv 1 (\mathit{mod}\ 4) \\ 1, & \dim Y \equiv -1 (\mathit{mod}\ 4) \end{cases}.$$

Chapter 3

A survey on Sasaki-Einstein manifolds

3.1 Sasakian structures and Einstein metrics

3.1.1 Symplectic manifolds and contact structures

A symplectic manifold (M, ω) is a manifold M with a symplectic form ω. In order for a 2-form ω to be a symplectic form, it satisfies

i. ω is closed. i.e. $d\omega = 0$.

ii. ω is non-degenerate. i.e. for any tangent vector v, $\omega(v, w) \neq 0$ for some tangent vector w.

M is necessarily even dimensional because a real skew-symmetric matrix of odd dimension must have a nontrivial kernel. Any differentiable function H on a symplectic manifold (M, ω) is called a Hamiltonian. Also, if a vector field X_H satisfies $dH = \omega(X_H, -)$, then it is called a Hamiltonian vector field, and its integral curve is called a Hamiltonian flow of H. To understand how to get to these definitions, let us consider the following example:

For a Riemannian manifold (N, g), we will determine *geodesics* on N. First, use the variations of energy method in [10]. For a curve $c : [0, a] \to N$ in N, denote its length by $L(c)$ and define the energy by $E(c) = \int_0^a \left|\frac{dc}{dt}\right|^2$. By the Schwarz inequality, we can easily see that $L(c)^2 \leqslant aE(c)$ and equality occurs iff t is proportional to arc length. There are theorems in [10] that present characterizations of a geodesic.

Lemma 3.1.1. *Let $p, q \in N$ and let $\gamma : [0, a] \to N$ be a minimizing geodesic joining p and q. Then, for all curves $c : [0, a] \to N$ joining p and q,*

$$E(\gamma) \leqslant E(c)$$

with equality holding if and only if c is a minimizing geodesic.

For a variation $f : (-\epsilon, \epsilon) \times [0, a] \to N$ of c, define the *energy function* $E(s)$ by

$$E(s) = \int_0^a \left| \frac{\partial f}{\partial t}(s,t) \right|^2 dt, \quad s \in (-\epsilon, \epsilon).$$

Proposition 3.1.2. *A piecewise differentiable curve $c : [0, a] \to N$ is a geodesic if and only if, for every proper variation f of c, we have $\frac{dE}{ds}(0) = 0$.*

Now, let's put this in Hamiltonian dynamics. Define a Lagrangian function L on the tangent bundle TN by

$$L(x.v) = \frac{1}{2}\|v\|^2,$$

where $x \in N$, $v \in T_x N$. Then,

$$E(c) = 2 \int_0^a L\left(c, \dot{c}\right) dt,$$

and by the above variation of Energy, a geodesic is a solution of the Euler-Lagrangian equations i.e. it is the minimal path in the sense that $\int_0^a L\left(\gamma, \dot{\gamma}\right)$ is minimal for any variations. Refer to the first chapter of [23] for more details.

If we let $M = T^*N$ whose coordinates are $(q_1, \cdots, q_n, p_1, \cdots, p_n)$, M has a symplectic form $\omega = d\lambda$, $\lambda = \sum_j p_j dq_j$. The Legendre transformation turns L into a Hamiltonian H on M which is

$$H(q, p) = \frac{1}{2}\|p\|^2,$$

where $p \in T_q^* N$. By direct calculations, we can see that the Hamiltonian flows of H are actually geodesics of (N, g). i.e. the Hamiltonian equation of H is nothing but the geodesic equation of (N, g). Apparently, this Hamiltonian H, or the Lagrangian L, is the kinetic energy of a particle, and hence it describes the dynamics of free particles. By varying the Hamiltonians, we can investigate a lot of dynamics on a Riemannian manifold in this manner.

Every symplectic manifold admits a compatible almost complex structure. In other words, for a symplectic manifold (M, ω), there exists an almost complex structure J on M such that $\langle v, w\rangle = \omega(v, Jw)$ defines a Riemannian metric on M, and $\langle v, w\rangle = \langle Jv, Jw\rangle$. There is an important theorem about it in [23].

Proposition 3.1.3. *Let M be a 2n-dimensional manifold.*

(i) For each nondegenerate 2-form ω on M there exists an almost complex structure J which is compatible with ω. The space $\mathcal{J}(M, \omega)$ of such almost complex structure is contractible.

(ii) For each almost complex structure J on M there exists a nondegenerate 2-form ω which is compatible with J. The space of such forms is contractible.

In case a symplectic manifold (M, ω) admits a compatible almost complex structure J, which is integrable, then we call (M, ω) is a Kähler manifold. Actually, this definition is equivalent to the notion in algebraic geometry.

Example 3.1.4. *Consider $\mathbb{R}^{2n} \cong \mathbb{C}^n$ with coordinates $(z_1, \cdots, z_n)$. It is an easy fact that $(\mathbb{R}^{2n}, \omega_0, J_0)$ is a Kähler manifold, where*

$$\omega_0 = \frac{\sqrt{-1}}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j,$$

and J_0 is the usual multiplication by $\sqrt{-1}$. If we write $z_j = x_j + \sqrt{-1} y_j$, then

$$\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j,$$

$$J_0\left(\frac{\partial}{\partial x_j}\right)=\frac{\partial}{\partial y_j},\ J_0\left(\frac{\partial}{\partial y_j}\right)=-\frac{\partial}{\partial x_j}.$$

Contact geometry is the odd-dimensional analogue of symplectic geometry. To equip an odd-dimensional manifold M with a symplectic form, we first consider a codimension 1 distribution $\mathcal{D}$. The following lemma in [15] informs when $\mathcal{D}$ is a kernel of a 1-form.

Lemma 3.1.5. *Locally, $\mathcal{D}$ can be written as the kernel of a differentiable 1-form η. It is possible to write $\mathcal{D}=\ker\eta$ with a 1-form η defined globally on all of M if and only if $\mathcal{D}$ is coorientable, which by definition means that the quotient line bundle $TM/\mathcal{D}$ is trivial.*

It contradicts to the Frobenius integrability condition if we want to use $d\eta$ as a symplectic form on $\mathcal{D}$. Choose two vectors in X, Y in $\mathcal{D}$ i.e. $\eta(X)=\eta(Y)=0$ and assume the Frobenius integrability. In order for $[X,Y]$ to be in $\mathcal{D}$,

$$0=\eta([X,Y])=X\eta(Y)-Y\eta(X)-d\eta([X,Y])=-d\eta([X,Y]).$$

Therefore $\eta\wedge d\eta=0$ everywhere on M and it doesn't go with the nondegeneracy requirement for a symplectic form. In this sense, contact structures must be the exact opposite of the integrable distributions. The following are definitions and examples in [15].

Definition 3.1.6. Let M be a manifold of odd dimension 2n+1. A contact structure is a maximally non-integrable hyperplane field $\mathcal{D}=\ker\eta$, that is, the defining differential 1-form η is required to satisfy

$$\eta\wedge(d\eta)^n\neq 0.$$

Such a 1-form η is called a contact form. The pair (M,η) is called a contact manifold.

There is another fundamental concept in contact geometry.

Definition 3.1.7. Associated with a contact form η one has the Reeb vector field $\mathcal{R}_\eta$, uniquely defined by the equations

(i) $d\eta\left(\mathcal{R}_\eta, -\right) \equiv 0$,

(ii) $\eta\left(\mathcal{R}_\eta\right) \equiv 1$.

Clearly, $\mathfrak{L}_{\mathcal{R}_\eta}\eta = 0$ because $\mathfrak{L}_{\mathcal{R}_\eta}\eta = d\left[\eta\left(\mathcal{R}_\eta\right)\right] + d\eta\left(\mathcal{R}_\eta, -\right) = 0$. Similarly, $\mathfrak{L}_{\mathcal{R}_\eta} d\eta = 0$.

Example 3.1.8. *On $\mathbb{R}^{2n+1}$ with Cartesian coordinates $\left(x_1, y_1, \cdots, x_n, y_n, z\right)$, the 1-form $\eta = dz + \sum\limits_{j=1}^{n} x_j dy_j$ is a contact form and its Reeb vector field $\mathcal{R}_\eta$ is ∂_z.*

Example 3.1.9. *Let $\left(x_1, y_1, \cdots, x_{n+1}, y_{n+1}\right)$ be Cartesian coordinates on $\mathbb{R}^{2n+2}$. The 1-form $\eta = \frac{1}{2}\sum\limits_{j=1}^{n+1}\left(x_j dy_j - y_j dx_j\right)$ restricted to the unit sphere S^{2n+1} is a contact form on S^{2n+1} and its Reeb vector field $\mathcal{R}_\eta$ is*

$$\mathcal{R}_\eta = J\left(\partial_r\right) = J\left(x_j\partial_{x_j} + y_j\partial_{y_j}\right) = x_j\partial_{y_j} - y_j\partial_{x_j}.$$

Example 3.1.10. *For a Riemannian manifold N, the hyperspace $T_1^*N \subset T^*N$ consisting of unit cotangent vectors of T^*N is a contact manifold with the contact form $\lambda = \sum_j p_j dq_j$. Its Reeb vector field generates the geodesic flows.*

Since the Reeb vector field $\mathcal{R}$ is never zero, it determine a 1-dimensional foliation $\mathcal{F}_\mathcal{R}$ on (M, η) called the *characteristic foliation.* Denote the trivial line bundle consisting of vectors tangent to $\mathcal{R}$ by $L_\mathcal{R}$. Then $TM = \mathcal{D} \oplus L_\mathcal{R}$. There are interesting facts about characteristic foliations presented in [6].

Definition 3.1.11. The characteristic foliation $\mathcal{F}_\mathcal{R}$ is said to be quasi-regular if there is a positive integer k such that each point has a foliated coordinate chart (U, x) such that each leaf of $\mathcal{F}_\mathcal{R}$ passes through U at most k times. If $k = 1$ then the foliation is called regular. If $\mathcal{F}_\mathcal{R}$ is not quasi-regular, it is said to be irregular.

Theorem 3.1.12. *Let (M, η) be a regular compact contact manifold. Then M is the total space of a principal circle bundle $\pi : M \to \mathcal{Z}$ over the space of leaves $\mathcal{Z} = M/\mathcal{F}_\mathcal{R}$. Furthermore, $\mathcal{Z}$ is a compact symplectic manifold with symplectic form Ω, $[\Omega] \in H^2\left(\mathcal{Z}, \mathbb{Z}\right)$, and η is a connection form on the bundle with curvature $d\eta = \pi^*\Omega$.*

3.1.2 Almost contact structures and Sasakian structures

In order to get to the notion of Sasakian structures, we start with a definition with weaker requirements than contact structures.

Definition 3.1.13. An almost contact structure on a differentiable manifold M is a triple $(\mathcal{R}, \eta, \Phi)$, where Φ is a tensor field of type (1,1), $\mathcal{R}$ is a vector field, and η is a 1-form which satisfy

$$\eta\left(\mathcal{R}\right) = \mathbb{1},\ \Phi \circ \Phi = -\mathbb{1} + \mathcal{R} \otimes \eta,$$

where $\mathbb{1}$ is the identity endomorphism on TM. A smooth manifold with such a structure is called an almost contact manifold.

Easy identities : $\Phi\left(\mathcal{R}\right) = 0, \eta \circ \Phi = 0$. If an almost contact manifold M admits a compatible metric g i.e. for all tangent vectors X, Y

$$g\left(\Phi X, \Phi Y\right) = g\left(X, Y\right) - \eta\left(X\right)\eta\left(Y\right),$$

then we say that M has an almost contact metric structure. Easily

$$g\left(X, \mathcal{R}\right) = \eta\left(\mathcal{R}\right),$$

by $Y = \mathcal{R}$.

Now, define a contact metric structure. Given a contact manifold (M, η), a Riemannian metric g is an associated metric if $\eta\left(X\right) = g\left(X, \mathcal{R}_\eta\right)$ and there exists a (1,1)-tensor field Φ such that

$$\Phi = -\mathbb{1} + \mathcal{R}_\eta \otimes \eta,\ d\eta\left(X, Y\right) = g\left(X, \Phi Y\right).$$

Such $(M, \mathcal{R}, \eta, \Phi, g)$ is called a contact metric manifold.

An easy but important fact is that the integral curves of $\mathcal{R}$ are geodesics in a contact metric manifold because

$$\begin{aligned}\left(\mathfrak{L}_{\mathcal{R}}\eta\right)\left(X\right) =& \mathcal{R}\eta\left(X\right) - \eta\left(\left[\mathcal{R}, X\right]\right) \\ =& \mathcal{R}g\left(X, \mathcal{R}\right) - g\left(\left[\mathcal{R}, X\right], \mathcal{R}\right)\end{aligned}$$

$$
\begin{aligned}
&= g\left(\nabla_{\mathcal{R}} X, \mathcal{R}\right) + g\left(X, \nabla_{\mathcal{R}} \mathcal{R}\right) - g\left(\nabla_{\mathcal{R}} X - \nabla_X \mathcal{R}, \mathcal{R}\right) \\
&= g\left(X, \nabla_{\mathcal{R}} \mathcal{R}\right) + g\left(\nabla_X \mathcal{R}, \mathcal{R}\right) \\
&= g\left(X, \nabla_{\mathcal{R}} \mathcal{R}\right) + \frac{1}{2} X g\left(\mathcal{R}, \mathcal{R}\right) \\
&= g\left(X, \nabla_{\mathcal{R}} \mathcal{R}\right) = 0,
\end{aligned}
$$

because $\mathfrak{L}_{\mathcal{R}} \eta = 0$, and $g\left(\mathcal{R}, \mathcal{R}\right) \equiv 1$. Therefore $\nabla_{\mathcal{R}} \mathcal{R} = 0$.

Now we consider the cone over an almost contact manifold to produce a new symplectic manifold. Let $(M, \mathcal{R}, \eta, \Phi)$ be an almost contact manifold and consider the cone $C\left(M\right) = M \times \mathbb{R}^+$. The following propositions in [6] inform how to build a symplectic cone.

Proposition 3.1.14. *Let η be a 1-form on the manifold M. Then η defines a contact structure on M if and only if the 2-form $d\left(r^2 \eta\right)$ defines a symplectic structure on the cone $C\left(M\right) = M \times \mathbb{R}^+$.*

Also,

Proposition 3.1.15. *There is a one-to-one correspondence between the contact metric structures $(\mathcal{R}, \eta, \Phi, g)$ on M and almost Kähler structures*

$$\left(dr^2 + r^2 g, d\left(r^2 \eta\right), J\right)$$

on $C\left(M\right)$.

Further, define an almost complex structure J on $C\left(M\right)$ by

$$J\left(X, f\frac{d}{dr}\right) = \left(\Phi X - f\mathcal{R}, \eta\left(X\right)\frac{d}{dr}\right),$$

where X is tangent to M, r the coordinate of $\mathbb{R}^+$, and f is a smooth function on $C\left(M\right)$. It really is an almost complex structure because

$$
\begin{aligned}
J^2\left(X, f\frac{d}{dr}\right) &= J\left(\Phi X - f\mathcal{R}, \eta\left(X\right)\frac{d}{dr}\right) \\
&= \left(\Phi^2 X - f\Phi\mathcal{R} - \eta\left(X\right)\mathcal{R}, \eta\left(\Phi X - f\mathcal{R}\right)\frac{d}{dr}\right)
\end{aligned}
$$

$$= \left(-X + \eta(X)\mathcal{R} - f\Phi\mathcal{R} - \eta(X)\mathcal{R}, -f\eta(\mathcal{R})\frac{d}{dr}\right)$$
$$= -\left(X, f\frac{d}{dr}\right).$$

Now, if J is integrable, we say that $(M, \mathcal{R}, \eta, \Phi)$ is normal. Using the Nijenhuis torsion, we look into this normality condition. Simple computations in [5] show that

$$\begin{aligned}
&[J, J]((X, 0), (Y, 0)) \\
&\quad = \left([\Phi, \Phi](X, Y) + 2d\eta(X, Y)\mathcal{R}, ((\mathfrak{L}_{\Phi X}\eta)(Y) - (\mathfrak{L}_{\Phi Y}\eta)(X))\frac{d}{dr}\right) \\
&[J, J]\left((X, 0), \left(0, \frac{d}{dr}\right)\right) = \left((\mathfrak{L}_{\mathcal{R}}\Phi)X, ((\mathfrak{L}_{\mathcal{R}}\eta)X)\frac{d}{dr}\right).
\end{aligned}$$

Therefore, we need to decompose it into four tensors $N^{(1)}, N^{(2)}, N^{(3)}$ and $N^{(4)}$, where

$$\begin{aligned}
N^{(1)}(X, Y) &= [\Phi, \Phi](X, Y) + 2d\eta(X, Y)\mathcal{R}, \\
N^{(2)}(X, Y) &= (\mathfrak{L}_{\Phi X}\eta)(Y) - (\mathfrak{L}_{\Phi Y})(X), \\
N^{(3)}(X) &= (\mathfrak{L}_{\mathcal{R}}\Phi)(X), \\
N^{(4)}(X) &= (\mathfrak{L}_{\mathcal{R}}\eta)(X).
\end{aligned}$$

Of course the almost contact structure is normal if and only if these four tensors vanish. There are relevant theorems listed in chapter 6 of [5].

Theorem 3.1.16. *For an almost contact structure the vanishing of $N^{(1)}$ implies the vanishing of $N^{(2)}, N^{(3)}$ and $N^{(4)}$.*

Theorem 3.1.17. *For a contact metric structure, $N^{(2)}$ and $N^{(4)}$ vanish. Moreover, $N^{(3)}$ vanishes if and only if $\mathcal{R}$ is a Killing vector field.*

Proof.

$$\begin{aligned}
0 = \left(d^2\eta\right)(\mathcal{R}, X, Y) &= \mathcal{R}d\eta(X, Y) - d\eta([\mathcal{R}, X], Y) - d\eta(X, [\mathcal{R}, Y]) \\
&= \mathcal{R}g(X, \Phi Y) - g([\mathcal{R}, X], \Phi Y) - g(X, \Phi([\mathcal{R}, Y]))
\end{aligned}$$

$$
\begin{aligned}
&=\mathcal{R}g\left(X,\Phi Y\right)-g\left(\left[\mathcal{R},X\right],\Phi Y\right)-g\left(X,\left[\mathcal{R},\Phi Y\right]\right)\\
&+g\left(X,\left[\mathcal{R},\Phi Y\right]\right)-g\left(X,\Phi\left(\left[\mathcal{R},Y\right]\right)\right)\\
&=\left(\mathfrak{L}_{\mathcal{R}}g\right)\left(X,\Phi Y\right)+g\left(X,\left(\mathfrak{L}_{\mathcal{R}}\Phi\right)Y\right).
\end{aligned}
$$

Therefore $\mathfrak{L}_{\mathcal{R}}g=0$ if and only if $\mathfrak{L}_{\mathcal{R}}\Phi=0$. □

A contact metric structure for which $\mathcal{R}$ is Killing is called a K-contact structure. Now, we are ready to define Sasakian structures.

Definition 3.1.18. A normal contact metric structure on M is called a Sasakian structure and M is called a Sasakian manifold.

Alternatively,

Definition 3.1.19. A normal contact metric structure on $(M,\mathcal{R},\eta,\Phi,g)$ is Sasakian if its metric cone $\left(C\left(M\right),dr^2+r^2g,d\left(r^2\eta\right),J\right)$ is Kähler.

There is more to say about the leaf space of the foliation $\mathcal{F}_{\mathcal{R}}$ with K-contact or Sasakian structures. See the following theorem in chapter 7 of [6].

Theorem 3.1.20. *Let $(M,\mathcal{R},\eta,\Phi,g)$ be a quasi-regular K-contact manifold with compact leaves. Then*

i. The space of leaves $M/\mathcal{F}_{\mathcal{R}}$ is an almost Kähler orbifold $\mathcal{Z}$ such that the canonical projection $\pi:M\to M/\mathcal{F}_{\mathcal{R}}$ is an orbifold Riemannian submersion.

ii. If $\mathcal{F}_{\mathcal{R}}$ is regular, then the circle action is free and M is the total space of a principal S^1-bundle over an almost Kähler, hence symplectic manifold defining an integral class $[\omega]\in H^2\left(M/\mathcal{F}_{\mathcal{R}},\mathbb{Z}\right)$.

iii. $(M,\mathcal{R},\eta,\Phi,g)$ is Sasakian if and only if $(M/\mathcal{F}_{\mathcal{R}},\omega)$ is Kähler.

In this respect, Sasakian structures may be viewed as odd-dimensional analogues of Kähler structures.

Example 3.1.21. *The easiest example of Sasakian manifolds is the odd dimensional sphere $M=S^{2n+1}$, because its metric cone is $\mathbb{R}^{2n+2}\backslash\{\mathbf{0}\}\cong\mathbb{C}^{n+1}\backslash\{\mathbf{0}\}$, which is the easiest example of Kähler manifolds. If we give*

$\eta = \frac{1}{2}\sum_{j=1}^{n+1}(x_j dy_j - y_j dx_j)$ and $\mathcal{R} = x_j\partial_{y_j} - y_j\partial_{x_j}$ on M, then its space of leaves is the complex projective space $\mathbb{C}P^n$ and $\pi : M \to M/\mathcal{F}_{\mathcal{R}}$ is the Hopf-fibration.

Example 3.1.22. *More generally, suppose we have a projective manifold X embedded in $\mathbb{C}P^n$ and consider its affine cone $C(X)$ in $\mathbb{C}^{n+1}$. Then its link $L = C(X) \cap S^{2n+1}(1)$ is a regular Sasakian manifold.*

Example 3.1.23. *Brieskorn manifolds are also important examples of Sasakian manifolds. Let $P(z) = z_0^{a_0} + \cdots + z_n^{a_n}$ be a Brieskorn polynomial, and $V(a_0, \cdots, a_n)$ its zero set in $\mathbb{C}^{n+1}$. The link $\Sigma(a_0, \cdots, a_n) = V(a_0, \cdots, a_n) \cap S^{2n+1}(1)$ is called a Brieskorn manifold, and it turns out to be Sasakian, not necessarily regular one. It is explained in §6.7 in [5].*

K-contact and Sasakian manifolds have many nice properties regarding curvatures. To develop useful formulas, the tensor $N^{(3)}$ plays an important role. The following lemma in chapter 7 of [6] provides the characteristic of $N^{(3)}$.

Lemma 3.1.24. *For a contact metric structure we have*

i. *$N^{(3)}$ is a symmetric endomorphism in the sense that*

$$g\left(N^{(3)}(X), Y\right) = g\left(N^{(3)}(Y), X\right).$$

ii. $\nabla\mathcal{R} = -\Phi - \frac{1}{2}\Phi N^{(3)}$.

iii. $\Phi \circ N^{(3)} + N^{(3)} \circ \Phi = 0$.

iv. $N^{(3)}(\mathcal{R}) = 0,\ \eta \circ N^{(3)} = 0$.

v. $\mathrm{tr} N^{(3)} = 0$.

The following are useful formulas regarding curvatures listed in chapter 7 of [5]. Verifications are omitted because they are too long and tricky.

Theorem 3.1.25. *A contact metric manifold M^{2n+1} is K-contact if and only if $\mathrm{Ric}(\mathcal{R}) = 2n$.*

Theorem 3.1.26. *A contact metric manifold is K-contact if and only if the sectional curvature of all plane sections containing $\mathcal{R}$ are equal to 1. Moreover, on a K-contact manifold,*

$$R_{X\mathcal{R}}\mathcal{R} = X - \eta(X)\mathcal{R}.$$

Proposition 3.1.27. *A contact metric structure is Sasakian if and only if*

$$R_{XY}\mathcal{R} = \eta(Y)X - \eta(X)Y.$$

Proposition 3.1.28. *Let (M^{2n+1}, g) be a Riemannian manifold admitting a unit Killing vector field $\mathcal{R}$ such that $R_{X\mathcal{R}}\mathcal{R} = X$ for X orthogonal to $\mathcal{R}$. Then M is a K-contact manifold. In this case, if*

$$R_{X\mathcal{R}}\mathcal{R} = g(\mathcal{R}, Y)X - g(X, \mathcal{R})Y,$$

then M^{2n+1} is Sasakian.

3.1.3 General relativity, Einstein manifolds

For a long period of time, classical physics is believed to obey the principle of relativity. That is, there should be equivalence between different descriptions based on different inertial frames that explain one physical phenomenon. Galileo stated his principle about relativity by the following 3 rules:

1. Inertial frames exist.

2. Physical laws take the same form in all inertial frames.

3. All inertial frames share a common time.

In Galilean relativity, time was absolute. However, it had been known to many physicists that the Galilean relativity was incompatible with electrodynamics. To remedy this defect, Albert Einstein introduced the special relativity theory in 1905. He took the first two postulates in Galilean relativity but the third one was replaced by *"All inertial frames share a common speed of light"*. In other words, simultaneity is meaningless in special relativity.

Special relativity can be succinctly described by the Lorentz transformation. Suppose we have one inertial frame with the coordinate (t, x, y, z), and another one with $(\bar{t}, \bar{x}, \bar{y}, \bar{z})$ that slides along the x-axis at constant speed v. Their origins coincide when the clock starts i.e. when $t = 0$. According to special relativity the two coordinates are linked by the following matrix form:

$$\begin{pmatrix} \bar{t} \\ \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix},$$

where $\gamma = \frac{1}{\sqrt{1-v^2}}$ (always assume the speed of light c equals one).

By fixing $\bar{x}$, we get $\Delta\bar{t} = \frac{1}{\gamma}\Delta t$, which explains the time dilation and if we fix $\bar{t}$, then the Lorentz contraction $\Delta\bar{x} = \gamma\Delta x$ occurs. On the other hand, the Lorentz transformation preserves the quantity

$$-(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \,,$$

which motivates *Minkowski metrics*.

Einstein also asserted that gravity and accelerated motion should be effectively indistinguishable. And by applying special relativity to one inertial frame that moves at accelerated speed with respect to another, he showed that warping or curving of space and time must exist in accelerated motions. According to Einstein, gravity is the warping of spacetime and general relativity is a theory that describes how warped the spacetime is in the vicinity of given mass. The spacetime is assumed to be a four dimensional manifold with a Minkowski metric g_{ab} - that is, g_{ab} is a nondegenerate symmetric bilinear form on the tangent bundle with one negative and three positive eigenvalues. The formula that describes the relation between gravity and g_{ab} is called the Einstein field equations(EFE):

$$G_{ab} + \Lambda g_{ab} = 8\pi G T_{ab},$$

where G_{ab} is the Einstein tensor, Λ is the cosmological constant, G is the Newton constant, and T_{ab} is stress-energy tensor generated by matters. The

Einstein tensor G_{ab} is defined to be $\mathrm{Ric}_{ab} - \frac{1}{2}Rg_{ab}$, where Ric_{ab} is the Ricci curvature tensor and R is the scalar curvature.

Even Einstein didn't believe that it was possible to solve the EFE with explicit solutions because it is a non-linear system of PDE's that does not fall into any classical categories, (e.g. elliptic, parabolic, hyperbolic etc.). However, a little after the publication of general relativity, K. Schwarzschild found one exact solution under a lot of symmetries and good conditions like $\Lambda = 0$, $T_{ab} = 0$. It is called Schwarzschild metric, which is

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}},$$

where M is the mass.

In case $T_{ab} = 0$, the EFE's

$$G_{ab} + \Lambda g_{ab} = 0$$

are referred to as the vacuum field equations, and their solutions are called *Einstein manifolds*. At this moment let us take a look at the definition stated in math textbooks, say [4] for example.

Definition 3.1.29. A pseudo-Riemannian manifold (M, g) is Einstein if there exists a real constant λ such that $\mathrm{Ric}\,(X, Y) = \lambda g\,(X, Y)$ for each X, Y in T_xM and each x in M.

A pseudo-Riemannian manifold means a manifold with a non-degenerate symmetric bilinear form g. It doesn't have to be positive definite as long as it keeps its signature consistent all over the points in M like the Minkowski metric. If we recall the definition of the Einstein tensor, the vacuum field equation is actually

$$\mathrm{Ric}_{ab} = (R - \Lambda)\, g_{ab}$$

and of course R could vary along the manifold. But there is a useful fact in [4] concerning it.

Theorem 3.1.30. *Assume $n \geqslant 3$. Then an n-dimensional pseudo-Riemannian manifold M is Einstein if and only if, for each x in M, there exists a constant λ_x such that $Ric_x = \lambda_x g_x$.*

Proof. Assume $\mathrm{Ric}_x = \lambda_x g_x$. By taking the trace on both sides we immediately get $R_x = n\lambda_x$ for all x. Now,

$$\sum_l \left(\nabla_{X_l} \sum_j R_{jk} g^{jl}\right) = \sum_l \left(\nabla_{X_l} \sum_j \lambda_x g_{jk} g^{jl}\right) = \nabla_{X_k} \lambda_x.$$

But one variation of the Bianchi identity is

$$\sum_l \nabla_{X_l} \sum_j R_{jk} g^{jl} = \frac{1}{2} \nabla_{X_k} R.$$

$\therefore\ \lambda - \frac{1}{2}R$ is constant, or $\left(1 - \frac{n}{2}\right)\lambda$ is constant. $\therefore\ \lambda$ is constant unless $n = 2$. $\square$

The 2-dimensional case is different. We can find the following facts in [4].

Proposition 3.1.31. *A 2 or 3-dimensional pseudo-Riemannian manifold is Einstein if and only if it has constant curvature.*

Theorem 3.1.32. *Any 2-dimensional manifold admits a complete metric with constant curvature.*

In this regard, Einstein manifold has a meaning as a physics term. From a mathematical point of view, Einstein metrics are also *good metrics* in a different sense. We need to consider the moduli space of Riemannian metrics and the Hilbert-Einstein action to figure out what that means.

Let M be an n-dimensional compact manifold without boundary and $\mathcal{M}$ the space of all Riemannian metrics on M. The Hilbert-Einstein action, which is also known as the total scalar curvature, is a functional defined on $\mathcal{M}$ by

$$S(g) = \int_M R_g d\mathrm{vol}_g.$$

To get critical points of S, we use the variation principle. Let h be any smooth symmetric tensor of type (0,2) on M, and set

$$g(t) = g + th$$

for $t \in (-\epsilon, \epsilon)$ so that $g(t)$ is a curve on $\mathcal{M}$. Let's use an upper dot for $\frac{d}{dt}\big|_{t=0}$. Because $R = \sum_{ij} \mathrm{Ric}_{ij} g^{ij}$,

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} R(t) &= \sum_{ij} \frac{d}{dt}\Big|_{t=0} \left(\mathrm{Ric}_{ij}(t)\, g^{ij}(t)\right) \\ &= \sum_{ij} \left(\frac{d}{dt}\Big|_{t=0} \mathrm{Ric}_{ij}(t)\right) g^{ij}(0) + \sum_{ij} \mathrm{Ric}_{ij}(0) \left(\frac{d}{dt}\Big|_{t=0} g^{ij}(t)\right).\end{aligned}$$

$\mathrm{Ric}_{ij}(0)$ is the Ricci curvature of g so just denote it by Ric_{ij}. On the other hand,

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} \left(\sum_j g^{ij}(t)\, g_{jl}(t)\right) &= \sum_j \left(\dot{g}^{ij} g_{jl}(0) + g^{ij}(0)\, \dot{g}_{jl}\right) \\ &= \sum_j \left(\dot{g}^{ij} g_{jl} + g^{ij} h_{jl}\right) \\ &= \frac{d}{dt}\Big|_{t=0} \delta_{il} = 0.\end{aligned}$$

$$\therefore \frac{d}{dt}\Big|_{t=0} g^{ij}(t) = -g^{ik} h_{kl} g^{jl}.$$

Recall that

$$\mathrm{Ric}_{ij} = \sum_\alpha \left\{ \Gamma^\alpha_{ij,\alpha} - \Gamma^\alpha_{\alpha i,j} + \sum_\beta \left(\Gamma^\alpha_{\alpha\beta} \Gamma^\beta_{ji} - \Gamma^\alpha_{j\beta} \Gamma^\beta_{\alpha i} \right) \right\},$$

where

$$\Gamma^\alpha_{ij} = \frac{1}{2} \sum_\beta g^{\alpha\beta} \left(g_{i\beta,j} + g_{j\beta,i} - g_{ij,\beta} \right).$$

By direct computations

$$\frac{d}{dt}\Big|_{t=0} R_{ij}(t) = \sum_\alpha \left(\partial_\alpha \dot{\Gamma}^\alpha_{ij} - \partial_j \dot{\Gamma}^\alpha_{i\alpha} + \sum_\beta \dot{\Gamma}^\alpha_{\alpha\beta} \Gamma^\beta_{ij} + \sum_\beta \Gamma^\alpha_{\alpha\beta} \dot{\Gamma}^\beta_{ij} - \sum_\beta \dot{\Gamma}^\alpha_{j\beta} \Gamma^\beta_{i\alpha} - \sum_\beta \Gamma^\alpha_{j\beta} \dot{\Gamma}^\beta_{i\alpha} \right).$$

Now, define a vector field $X = \sum_\alpha X_\alpha \frac{\partial}{\partial x_\alpha}$, where

$$X_\alpha = \sum_{ij} \left(g^{ij} \dot{\Gamma}_{ij}^{\alpha} - g^{i\alpha} \dot{\Gamma}_{ij}^{j} \right).$$

Then its divergence is

$$\begin{aligned}
\mathrm{div} X =& \sum_\alpha \left(\frac{\partial}{\partial_\alpha} \sum_{ij} \left(g^{ij} \dot{\Gamma}_{ij}^{\alpha} - g^{i\alpha} \dot{\Gamma}_{ij}^{j} \right) + \sum_\beta \Gamma_{\beta\alpha}^{\alpha} \sum_{ij} \left(g^{ij} \dot{\Gamma}_{ij}^{\beta} - g^{i\beta} \dot{\Gamma}_{ij}^{j} \right) \right) \\
=& \sum_{ij\alpha\beta} \left(\frac{\partial}{\partial_\alpha} \dot{\Gamma}_{ij}^{\alpha} g^{ij} - \dot{\Gamma}_{ij}^{\alpha} \Gamma_{\alpha\beta}^{i} g^{j\beta} - \dot{\Gamma}_{ij}^{\alpha} \Gamma_{\alpha\beta}^{j} g^{i\beta} \right) \\
&- \sum_{ij\alpha\beta} \left(\frac{\partial}{\partial_\alpha} \dot{\Gamma}_{ij}^{j} g^{i\alpha} - \dot{\Gamma}_{ij}^{j} \Gamma_{\alpha\beta}^{i} g^{\alpha\beta} - \dot{\Gamma}_{ij}^{j} \Gamma_{\alpha\beta}^{\alpha} g^{i\beta} \right) \\
&+ \sum_{ij\alpha\beta} \left(\dot{\Gamma}_{ij}^{\beta} \Gamma_{\beta\alpha}^{\alpha} g^{ij} - \dot{\Gamma}_{ij}^{j} \Gamma_{\beta\alpha}^{\alpha} g^{i\beta} \right) \\
=& \sum_{ij\alpha\beta} \left(\frac{\partial}{\partial_\alpha} \dot{\Gamma}_{ij}^{\alpha} - \dot{\Gamma}_{j\beta}^{\alpha} \Gamma_{\alpha i}^{\beta} - \dot{\Gamma}_{i\beta}^{\alpha} \Gamma_{\alpha j}^{\beta} \right) g^{ij} \\
&- \sum_{ij\alpha\beta} \left(\frac{\partial}{\partial_j} \dot{\Gamma}_{i\alpha}^{\alpha} - \dot{\Gamma}_{\alpha\beta}^{\beta} \Gamma_{ij}^{\alpha} - \dot{\Gamma}_{i\beta}^{\beta} \Gamma_{\alpha j}^{\alpha} \right) g^{ij} \\
&+ \sum_{ij\alpha\beta} \left(\dot{\Gamma}_{ij}^{\beta} \Gamma_{\beta\alpha}^{\alpha} - \dot{\Gamma}_{i\beta}^{\beta} \Gamma_{j\alpha}^{\alpha} \right) g^{ij} \\
=& \sum_{ij} \sum_{\alpha\beta} \left(\frac{\partial}{\partial_\alpha} \dot{\Gamma}_{ij}^{\alpha} - \frac{\partial}{\partial_j} \dot{\Gamma}_{i\alpha}^{\alpha} + \dot{\Gamma}_{\alpha\beta}^{\beta} \Gamma_{ij}^{\alpha} + \dot{\Gamma}_{i\beta}^{\beta} \Gamma_{\alpha j}^{\alpha} - \dot{\Gamma}_{j\beta}^{\alpha} \Gamma_{\alpha i}^{\beta} - \dot{\Gamma}_{i\beta}^{\alpha} \Gamma_{\alpha j}^{\beta} \right) g^{ij} \\
=& \sum_{ij} \left. \frac{d}{dt} \right|_{t=0} \mathrm{Ric}_{ij} g^{ij}.
\end{aligned}$$

Also,

$$\left. \frac{d}{dt} \right|_{t=0} \sqrt{\det g(t)} = \frac{\left. \frac{d}{dt} \right|_{t=0} \det g(t)}{2\sqrt{\det g}} = \frac{1}{2} \mathrm{Tr} \left(g^{-1} h \right) \sqrt{\det g}.$$

Therefore,

$$\left. \frac{d}{dt} \right|_{t=0} S(g) = \int_M \sum_{ijkl} -\mathrm{Ric}_{ij} g^{ik} h_{kl} g^{jl} + \mathrm{div} X + \frac{1}{2} \mathrm{Tr} \left(g^{-1} h \right) d\mathrm{vol}_g$$

$$= \int_M \sum_{ijkl} -\mathrm{Ric}_{ij} g^{ik} h_{kl} g^{jl} + \frac{1}{2} \mathrm{Tr}\left(g^{-1}h\right) d\mathrm{vol}_g,$$

because M has no boundary. If we define

$$\langle A, B \rangle_g = \mathrm{Tr}\left(\left(g^{-1}A\right)\left(g^{-1}B\right)^t\right),$$

for two matrices A, B, then

$$\left.\frac{d}{dt}\right|_{t=0} S(g) = \int_M \left\langle h, -\mathrm{Ric}_g + \frac{1}{2}Rg \right\rangle_g d\mathrm{vol}_g.$$

Theferefore, g is a critical point of S if

$$\mathrm{Ric}_g = \frac{1}{2}Rg.$$

i.e. g is an Einstein metric on M.

Now, let $\mathcal{M}_1$ be the subset of $\mathcal{M}$ consisting of metrics of total volume one. The spaces $\mathcal{M}/\mathcal{D}$ and $\mathcal{M}_1/\mathcal{D}$, where $\mathcal{D}$ is the diffeomorphism group of M, behave like manifolds when we give $\mathcal{M}$ the compact open C^∞ topology. Refer to chapter 4. of [4] for more details.

3.2 Kähler-Einstein metrics

3.2.1 Einstein conditions in Kähler metrics

Sasaki-Einstein metrics are closely related to Kähler-Einstein metrics because the quotient spaces of the Reeb flow actions become Kähler-Einstein in regular cases. Also Kähler-Einstein metrics provide a good research area on its own. In fact, a lot of known examples of Einstein manifolds happen to be Kähler, and we will see the most elementary ones in this section.

In case of Kähler manifolds, the Einstein equations $\mathrm{Ric}_g = \lambda g$ can be modified more effectively. On a Kähler manifold (M, J, g, ω_g), the Ricci form ρ is defined as

$$\rho(X, Y) := \mathrm{Ric}_g(JX, Y),$$

where J is a complex structure on M and g is a Riemannian metric, and ω_g is its Kähler form. Actually, ρ is a (real) 2-form because for an orthonormal basis $\{e_j, Je_j\}_{\{j=1,2,\cdots,m\}}$

$$\begin{aligned}
\mathrm{Ric}_g(X,Y) &= \sum_j \langle R(X,e_j)Y, e_j\rangle + \sum_j \langle R(X,Je_j)Y, Je_j\rangle \\
&= \sum_j \langle R(e_j,X)e_j, Y\rangle + \sum_j \langle R(Je_j,X)Je_j, Y\rangle \\
&= \sum_j \langle R(e_j,X)Je_j, JY\rangle + \sum_j \langle JR(Je_j,X)e_j, Y\rangle \\
&= -\sum_j \langle R(X,e_j)Je_j, JY\rangle - \sum_j \langle R(Je_j,X)e_j, JY\rangle \\
&= \sum_j \langle R(e_j, Je_j,)X, JY\rangle .
\end{aligned}$$

Therefore,

$$\begin{aligned}
\rho(X,Y) =& \mathrm{Ric}_g(JX,Y) = \mathrm{Ric}_g(Y,JX) \\
&= -\sum_j \langle R(e_j, Je_j,)Y, X\rangle \\
&= -\sum_j \langle R(e_j, Je_j,)JY, YX\rangle = -\rho(Y,X)
\end{aligned}$$

Since Ric_g is a symmetric tensor, ρ is a (1,1)-form and the Einstein equations are rewritten as

$$\rho = \lambda\omega_g.$$

Of course, we may assume that $\lambda = 0$ or $\lambda = \pm 1$ by rescaling.

Now, let's consider the holomorphic tangent bundle $E = T'M$ over M. It is identified with the (real) tangent bundle TM over M because they have the same transition functions in common. With this identification, the Levi-Civita connection on TM becomes a Chern connection D of E - i.e. it is metric compatible and the (0,1)-part $D''$of D is equal to $\bar{\partial}$ operator - because M is Kähler. The curvature R of D is defined by $R = D^2$, which turns out to be the same notion of the curvature in the Levi-Civita connection. R is an End(E)-valued (1,1)-form and by specifying a local frame it has the matrix representation Ω which is

$$\Omega^t = d'' \left(d' H H^{-1} \right),$$

where H is a matrix representation of the hermitian metric with respect to a local frame. Since we know that

$$\rho(X,Y) = \sum_j \langle R\left(e_j, Je_j, \right) X, Y \rangle = \sum_j \langle R\left(X,Y\right) e_j, Je_j \rangle ,$$

ρ is equal to $\sqrt{-1}$ times the trace of Ω.

The determinant bundle $\det(E)$ of E has a hermitian metric $\det\left(g_{i\bar{j}}\right)$ whose Chern connection form is the trace of the connection form of D. And its curvature is also the trace of the curvature of D. Clearly, the Ricci form ρ is the curvature of this $\det(E)$ bundle and so

$$\rho = \sqrt{-1}\mathrm{trace}\left(\Omega\right) = \sqrt{-1}\bar{\partial}\partial \log \det\left(g_{i\bar{j}}\right).$$

Finally, by using the Chern-Weil formula, the first Chern class $c_1\left(M\right)$ is represented by the class of $\left[\frac{1}{2\pi}\rho\right]$. Therefore, the Einstein condition implies that

$$c_1\left(M\right) = \frac{1}{2\pi}\left[\rho\right] = \frac{\lambda}{2\pi}\left[\omega_g\right]$$

hence $c_1\left(M\right) = 0$ or $c_1\left(M\right)$ is (positive or negative) definite. Furthermore, when M is compact, the sign of λ is consistent even if we replace g by another Kähler-Einstein metric because $\left[\omega_g\right] \neq 0$. Let's consider some examples.

Example 3.2.1. *The m-dimensional complex space $\mathbb{C}^m$ and the complex tori $\mathbb{C}/L$*

$$L = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z},$$

$\omega_1, \omega_2 \in \mathbb{C}$ *have flat metric. Therefore they are Kähler-Einstein with $\lambda = 0$.*

Example 3.2.2. *The Fubini-Study metric ω_{FS} on the m-dimensional com-*

plex projective $\mathbb{C}P^m$ defined by

$$\omega_{FS} = \frac{\sqrt{-1}}{2}\partial\bar{\partial}\log\|z\|,$$

where $z = (z_0, \cdots, z_m)$ is the coordinate on $\mathbb{C}^{m+1}$, is Kähler-Einstein with $\lambda > 0$.

Example 3.2.3. *Consider $M = \{z \in \mathbb{C}^m : \|z\| < 1\}$ and the Bergman metric*

$$\omega_B = -\frac{\sqrt{-1}}{2}\partial\bar{\partial}\log\left(1 - \|z\|^2\right)$$

on M. It is a Kähler-Einstein metric with $\lambda < 0$.

Remark 3.2.4. *$M = \{z \in \mathbb{C}^m : \|z\| < 1\}$ admits Kähler-Einstein metrics of all kinds with $\lambda = 0, \lambda = \pm 1$, which are the flat, the Fubini-Study and the Bergman metric. This may not happen when M is compact.*

3.2.2 Calabi conjecture and Calabi-Yau manifolds

In the previous section, we saw that the Ricci curvature form is in the same class as $2\pi c_1(M)$ on a Kähler manifold M. The Calabi conjecture is the question whether the converse holds as well and in [31] Yau, S.T. proved that it is true if M is compact. In other words,

Theorem 3.2.5. *If (M, g, ω_g) is a compact Kähler manifold, and R is any (1,1)-form representing the first Chern class of M, then there exists a unique Kähler metric $\tilde{g}$ on M with Kähler form $\omega_{\tilde{g}}$ such that its Ricci form $\rho_{\tilde{g}}$ is equal to R and ω_g, $\omega_{\tilde{g}}$ are in the same class in $H^2(M, \mathbb{R})$.*

In this section we'll consider Calabi-Yau manifolds briefly after an overview of Yau's proof because the theorem ensures the existence of so called Calabi-Yau metrics.

Let's set a new metric

$$\tilde{g}_{\imath\bar{\jmath}} = g_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z_i \partial\bar{z}_j},$$

for some smooth function φ on M, so that $\tilde{g}_{i\bar{j}}$ is still positive definite. Then its Ricci form is

$$\sqrt{-1}\bar{\partial}\partial \log\det\left(g_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z_i \partial \bar{z}_j}\right).$$

Since R is in the class of $2\pi c_1(M)$,

$$R = \sqrt{-1}\bar{\partial}\partial \log\det\left(g_{i\bar{j}}\right) + \sqrt{-1}\bar{\partial}\partial F,$$

for some smooth function F on M. Therefore the question boils down to whether there exists a smooth function φ which satisfies

$$\bar{\partial}\partial \log\det\left(g_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z_i \partial \bar{z}_j}\right) = \bar{\partial}\partial \log\det\left(g_{i\bar{j}}\right) + \bar{\partial}\partial F,$$

or

$$\det\left(g_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z_i \partial \bar{z}_j}\right) = \det\left(g_{i\bar{j}}\right) e^F.$$

To use the continuity method, let's consider the following family of equations:

$$\det\left(g_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z_i \partial \bar{z}_j}\right)\det\left(g_{i\bar{j}}\right)^{-1} = \mathrm{vol}(M)\left[\int_M e^{tF} dV_g\right]^{-1} e^{tF} - (*)_t,$$

and define

$$S = \left\{t \in [0,1] : (*)_t \text{ has a solution in } C^{k+1,\alpha}(M)\right\}.$$

If we can show S is a nonempty connected set, then the existence of φ is guaranteed. At first, the trivial function $\varphi \equiv 0$ is the solution when $t = 0$, and hence S is nonempty.

To show that S is open, we need the inverse function theorem. First, define some subsets of the Banach spaces by

$$\theta = \left\{\varphi \in C^{k+1,\alpha}(M) : 1 + \varphi_{i\bar{i}} > 0, \forall i \text{ and } \int_M \varphi = 0\right\},$$

and

$$B = \left\{\varphi \in C^{k-1,\alpha}(M) : \int_M f = \mathrm{vol}(M)\right\}.$$

Apparently, θ is an open subset of a hypersubspace in $C^{k+1,\alpha}$ and B is a hyperplane of $C^{k-1,\alpha}$. Define a function $G : \theta \longrightarrow B$ so that

$$\varphi \mapsto \det\left(g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}\right) \det\left(g_{i\bar{j}}\right)^{-1},$$

whose differential is

$$dG_{\varphi_0} = \det\left(g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}\right) \det\left(g_{i\bar{j}}\right)^{-1} \Delta_{\varphi_0},$$

at φ_0. It is well-known that the equation

$$\Delta_{\varphi_0}\varphi = f$$

has a unique solution whenever $\int_M f = 0$. Consequently, G is a local diffeomorphism and hence S is open.

It is the most difficult part to show S is closed. In fact, the Monge-Ampére equations are not linear, and this makes it very tough to obtain exact solutions. Instead, let's assume φ_q is a solution of $(*)_{t_q}$ and differentiate $(*)_{t_q}$ to get

$$LHS : \underbrace{\det\left(g_{1\bar{j}} + \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}\right) \sum_{i,j} {g'}^{i\bar{j}}}_{\text{linear operator}} \quad \underbrace{\frac{\partial^2}{\partial z_i \partial \bar{z}_j}\left(\frac{\partial \varphi_q}{\partial z_p}\right)}_{\text{solution}} \quad = \cdots$$

where g' is the inverse matrix of $\left(g_{1\bar{j}} + \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}\right)_{ij}$. Here we can regard $\frac{\partial \varphi_q}{\partial z_p}$ as a solution of the 2nd order linear partial differential equation. To show this linear operator is uniformly elliptic, Yau, S.T. verified the following proposition whose proof is extremely long.

Proposition 3.2.6. *Let M be a compact Kähler manifold with metric tensor $\sum_{i,\bar{j}} g_{i\bar{j}} dz_i \otimes d\bar{z}_j$. Let φ be a real-valued function in $C^4(M)$ such that $\int_M \varphi = 0$ and $\sum_{i,\bar{j}}\left(g_{1\bar{j}} + \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}\right) dz_i \otimes d\bar{z}_j$ defines another metric tensor on M. Suppose* $\det\left(g_{1\bar{j}} + \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}\right) \det\left(g_{1\bar{j}}\right)^{-1} = \exp(F)$. *Then there are positive constants C_1,*

C_2, C_3 and C_4 depending on $\inf_M F$*,* $\sup_M F$*,* $\inf_M \Delta F$ *and M such that* $\sup_M |\varphi| \leqslant C_1$*,* $\sup_M |\nabla \varphi| \leqslant C_1$*,* $0 < C_3 \leqslant 1 + \varphi_{i\bar{i}} \leqslant C_4$ *for all i.*

Also, if we can get the Hölder norm estimates of the coefficients in this elliptic operator, we can bound the Hölder norm of the solution as well. To do that, Yau, S.T. provided the following proposition:

Proposition 3.2.7. *With the same assumption except $\varphi \in C^5(M)$, there is an estimate of the derivatives $\varphi_{i\bar{j}k}$ in terms of $\sum_{i,\bar{j}} g_{i\bar{j}} dz_i \otimes d\bar{z}_j$,* $\sup_M |F|$*,* $\sup_M |\nabla F|$*,* $\sup_M \sup_i |F_{i\bar{i}}|$ *and* $\sup_M \sup_{i,j,k} |F_{i\bar{j}k}|$*.*

By using the Shauder estimates, we can say that

$$|\varphi_q|_{k+1,\alpha} \leqslant C,$$

and hence $\{\varphi_q\}$ converges in the $C^{k+1,\alpha}$-norm to a solution of $(*)_{t_0}$. Therefore, $t_q \longrightarrow t_0$ and S is closed.

Calabi-Yau manifolds are compact, Kähler manifolds that have trivial first Chern class. By convention, we exclude those with infinite fundamental groups. In fact, there are several different definitions for Calabi-Yau manifolds in use in the literature, but we stick with this one that is provided in [32]. Because of the Yau's theorem, Calabi-Yau manifolds always have Ricci-flat metrics, which are called Calabi-Yau metrics. Let's take a look at the following proposition in [17] at this point. With this proposition, we can define Calabi-Yau manifolds alternatively using the (restricted) holonomy groups.

Proposition 3.2.8. *Let (M, J, g) be a Kähler manifold. Then* $\mathrm{Hol}^0(g) \subset \mathrm{SU}(m)$ *if and only if g is Ricci-flat.*

Example 3.2.9. *Let X be a non-singular complete intersection of hypersurfaces H_j's of degrees d_j's $j = 1, \cdots, k$ in $\mathbb{C}P^m$. By the adjunction formula $c_1(X) = 0$ if $d_1 + \cdots + d_k = m + 1$. Also, because* $\dim_{\mathbb{C}} X = m - k$*, X is simply connected for $m - k \geqslant 2$.*

3.2.3 Kähler-Einstein metrics on del Pezzo surfaces

Now we will see how this Calabi-Yau theorem is related to the existence of Kähler-Einstein metrics on compact Kähler manifolds. Because the Ricci form represents the first Chern class $c_1(M)$, if M is a Kähler-Einstein manifold, $c_1(M)$ must be either 0 or (positive or negative) definite. Consider the case $c_1(M) = 0$ first. The Einstein equation $\rho = \lambda\omega_g$ in this case implies that

$$0 = c_1(M) = [\rho] = \lambda[\omega_g] \Rightarrow \lambda = 0,$$

because the Kähler form $[\omega_g]$ can't be 0 on a compact manifold. Therefore, we can see that the existence of the Kähler-Einstein metric is an easy consequence of the Calabi-Yau theorem.

Now, consider the cases with definite Chern classes. Assume $c_1(M) = \lambda[\omega_g]$, $\lambda = \pm 1$. Then $\rho = \lambda\omega + \sqrt{-1}\partial\bar{\partial}F$ for some $F \in C^\infty(M, \mathbb{R})$ because they are in the same class. We try to find a metric $\tilde{g}$ with the Kähler form $\tilde{\omega}$ such that $\tilde{\rho} = \lambda\tilde{\omega}$, where $\tilde{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ for some $\varphi \in C^\infty(M, \mathbb{R})$. Since

$$\begin{aligned}\tilde{\rho} - \rho =& \lambda\tilde{\omega} - \lambda\omega - \sqrt{-1}\partial\bar{\partial}F, \\ =& \lambda\sqrt{-1}\partial\bar{\partial}\varphi - \sqrt{-1}\partial\bar{\partial}F \\ =& \sqrt{-1}\bar{\partial}\partial \log\det\left(g_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z_i \partial\bar{z}_j}\right) - \sqrt{-1}\bar{\partial}\partial \log\det\left(g_{i\bar{j}}\right),\end{aligned}$$

we get

$$e^{F-\lambda\phi} = \frac{\det\left(g_{ij} + \frac{\partial^2\varphi}{\partial z_i\partial\bar{z}_j}\right)}{\det\left(g_{ij}\right)}$$

and apply the continuity method to this equation.

In the paper where Yau, S.T prove the Calabi-Yau theorem, he also shows the existence of a solution with negative λ. For example, suppose a solution φ to

$$\det\left(g_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z_i\partial\bar{z}_j}\right)\det\left(g_{i\bar{j}}\right)^{-1} = \exp\left(tF + \varphi\right), t > 0$$

achieves its maximum at z_0. Then, $\frac{\partial^2\varphi}{\partial z_i\partial\bar{z}_j}$ is negative definite, so $\det\left(g_{1\bar{j}} + \frac{\partial^2\varphi}{\partial z_i\partial\bar{z}_j}\right) \leqslant \det\left(g_{1\bar{j}}\right)$, and $\exp\left(tF + \varphi\right)(z_0) \leqslant 1$. Therefore, $\sup_M \varphi \leqslant \varphi(z_0) \leqslant -tF(z_0)$. Similarly, we can show that $\inf_M \varphi \geqslant -tF(z_1)$, for some z_1. Thus, we get a bound for $\|\varphi\|$.

The existence of a Kähler-Einstein metric fails when $c_1(M) > 0$ and there are some known sufficient conditions of existence and obstructions to it. Compact manifolds with positive first Chern classes are called Fano manifolds, and they are classified for 1,2,3 dimensional cases. For example, 2-dimensional Fano manifolds are diffeomorphic to $\mathbb{C}P^2$, $\mathbb{C}P^1 \times \mathbb{C}P^1$ or $\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$, where $1 \leqslant n \leqslant 8$. The surfaces $\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$ are also obtained by blowing up $\mathbb{C}P^2$ at n generic points, where "generic"means no three points are colinear, and no six points are in one quadratic curve in $\mathbb{C}P^2$.

Clearly, $\mathbb{C}P^2$ and $\mathbb{C}P^1 \times \mathbb{C}P^1$ are Kähler-Einstein manifolds because of the Fubini-Study metric. Also, it is known that $\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$ have Kähler-Einstein metrics for $n = 3, \cdots, 8$ while $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ and $\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$ don't.

The 2-dimensional Fano manifolds are also called del Pezzo surfaces but actually there are variations to define del Pezzo surfaces. Let's consider the definition in [12] for example:

Definition 3.2.10. A normal algebraic surface S is called a del Pezzo surface if its canonical sheaf ω_S is invertible, ω_S^{-1} is ample and all singularities are rational double points.

It admits rather mild singularities. And also we have a notion of degree in this definition:

Definition 3.2.11. The number $d = K_X^2 = N - 9$ is called the degree of a del Pezzo surface, where X is a minimal resolution of S and N is the number of blow-ups.

The following corollary gives the classification of (nonsingular) del Pezzo surface.

Corollary 3.2.12. *Assume that S is a nonsingular del Pezzo surface. Then $S \cong \mathbb{C}P^1 \times \mathbb{C}P^1$ or is obtained by blowing-up of a bubble cycle in $\mathbb{C}P^2$ of $\leqslant 8$ points.*

Let's go over briefly the automorphism groups of del Pezzo surfaces of degree $d = 6, 7, 8, 9$ for later use. Refer to chapter 8 in [12] for more detailed explanations.

1. $S = \mathbb{C}P^2$
$$\mathrm{Aut}(S) = PGL_3.$$

2. $S = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$

$$\mathrm{Aut}(S) = \mathbb{C}^2 \rtimes GL(2).$$

$\because$ Consider the subgroup of $\mathrm{Aut}\left(\mathbb{C}P^2\right)$ fixing a point. If we set $[1,0,0]$ fixed, then the invertible matrix is of the form

$$\begin{bmatrix} \varpi & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}, \varpi \neq 0,$$

and by Hartog's theorem, it is extended to the whole $S = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. By considering the action of nonzero diagonal matrices, we may assume that it is of the form

$$\begin{bmatrix} 1 & \alpha & \beta \\ 0 & & \\ 0 & \multicolumn{2}{c}{Q} \end{bmatrix}$$

, where $\alpha, \beta \in \mathbb{C}, Q \in GL(2,\mathbb{C})$. So, $[\alpha, \beta] \in \mathbb{C}^2$ and $Q \in GL(2,\mathbb{C})$ acts on $\mathbb{C}^2$ by

$$[\alpha, \beta] \mapsto [\alpha, \beta]\, Q,$$

to make a semidirect product group.

3. $S = \mathbb{C}P^2 \# \overline{2\mathbb{C}P^2}$

$$\mathrm{Aut}(S) = G \rtimes \mathbb{Z}_2,$$

where G is the subgroup of PGL_3 consists of invertible matrices of the form

$$\begin{bmatrix} 1 & 0 & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}.$$

$\because$ An element of $\mathrm{Aut}\left(\mathbb{C}P^2\right)$ that fixes two points, say $[1,0,0]$ and $[0,1,0]$, must be of the above form. But we may switch $[1,0,0]$ and $[0,1,0]$, so that $\mathrm{Aut}(S) = G \rtimes \mathbb{Z}_2$.

4. $S = \mathbb{C}P^2 \# \overline{3\mathbb{C}P^2}$

$$\mathrm{Aut}(S) = \left(\mathbb{C}^*\right)^2 \rtimes \left(S_3 \times \mathbb{Z}_2\right).$$

$\because$ An element of Aut $\left(\mathbb{C}P^2\right)$ that fixes three points $[1,0,0],[0,1,0]$ and $[0,0,1]$, is of the form,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{bmatrix}, \alpha, \beta \in \mathbb{C}^*.$$

And we may permute these three points $[1,0,0],[0,1,0]$ and $[0,0,1]$, which amounts to the permutation group S_3. Finally, there is a Cremona transformation φ that is undefined at $[1,0,0],[0,1,0]$ and $[0,0,1]$

$$\varphi : [X_0, X_1, X_2] \mapsto [X_1X_2, X_0X_2, X_0X_1],$$

which contributes to $\mathbb{Z}_2$.

In [26],Tian, G. provides a sufficient condition for the existence of a Kähler-Einstein metric with positive first Chern class and then applies it to the cases of $\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$, $n = 3, \ldots, 8$ in [27] with Yau, S.T. To prove the closedness when using the continuity method, they used the following notions:

$$P_G(M,g) = \left\{\phi \in C^2(M,\mathbb{R}) : \phi \text{ is } G \text{ invariant and } \omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\phi \geqslant 0\right\},$$

and

$$\alpha_G(M) = \sup\left\{\alpha | \exists C \text{ s.t. } \int_M e^{-\alpha(\phi(Z) - \sup_M \phi)} dV_g \leqslant C, \forall \phi \in P_G(M,g)\right\}$$

for a compact subgroup G of Aut(M). Denote $\alpha(M) = \alpha_G(M)$ for a trivial subgroup G.

We can easily deduce that $\alpha(M)$ is independent of the particular choice of g and that $\alpha_{G_1}(M) \leqslant \alpha_{G_2}(M)$ if $G_1 \leqslant G_2$.

There is a theorem to compute $\alpha_G(M)$ more easily.

Theorem 3.2.13. $\alpha_G(M) \geqslant \frac{1}{L_G(M)}$ *whenever M is Kähler and* $c_1(M) > 0$, *where*

$$L_G(M) = \sup_{z\in M}\left\{L_g(u,z) : u \text{ is } \begin{array}{c}\text{d-closed, positive (1,1)-current,}\\ G\text{-invariant, cohomological to } c_1(M)\end{array}\right\}.$$

Their main work is the following theorem:

Theorem 3.2.14. *For a compact Kähler manifold M with $c_1(M) > 0$, if $\alpha(M) > \frac{m}{m+1}$, where $m = \dim_{\mathbb{C}} M$, G is the maximal compact subgroup of* Aut(M)*, then M admits a Kähler-Einstein metric.*

And they estimate $\alpha_G(M)$ from below for some surfaces. For example, because $\mathbb{C}P^2\#\overline{3\mathbb{C}P^2}$ is a blowing-up of a Kähler manifold over a compact subset (3 points $[1,0,0]$, $[0,1,0]$ and $[0,0,1]$), it is also a Kähler manifold. G is generated by

$$\begin{pmatrix} e_1 & 0 & 0\\ 0 & e_2 & 0\\ 0 & 0 & e_3\end{pmatrix} \in PSL(3),\ \ |e_1|^2+|e_2|^2+|e_3|^2 = 1$$

and

$$\begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}, \begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} \in PSL(3).$$

$\alpha_G(M)$ is estimated to be $\geqslant 1$ so that we see $\mathbb{C}P^2\#\overline{3\mathbb{C}P^2}$ admits a Kähler-Einstein metric.

3.3 Sasaki-Einstein manifolds

3.3.1 Basic properties

The Einstein condition is more restrictive on metric cones than on usual Riemannian manifolds. Suppose $C(Y) = \mathbb{R}^+ \times Y$ has the cone metric $\tilde{g} = dr^2 + r^2 g$, where (Y,g) is the m-dimensional Riemannian base manifold. If we denote the Christoffel symbols of g and $\tilde{g}$ by $\Gamma^k_{ij}, \tilde{\Gamma}^k_{ij}$ respectively and name the coordinate r of $\mathbb{R}^+$ as the zeroth coordinate, then $\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij}$ holds for nonzero i, j, k's. Also, $\tilde{\Gamma}^0_{00} = \tilde{\Gamma}^0_{0j} = \tilde{\Gamma}^0_{i0} = \tilde{\Gamma}^k_{00} = 0$ and $\tilde{\Gamma}^0_{ij} = -rg_{ij}, \tilde{\Gamma}^k_{0j} = \frac{\delta_{kj}}{r}$. Similarly, if we denote the Ricci curvatures of $g, \tilde{g}$ by $R_{ij}, \tilde{R}_{ij}$, then $\tilde{R}_{00} = \tilde{R}_{i0} = \tilde{R}_{0j} = 0$ and

$$\tilde{R}_{ij} = R_{ij} - (m-1)\, g_{ij}$$

holds for nonzero i, j's. Therefore, if a cone metric $\tilde{g}$ is Einstein, then it must be Ricci-flat. Also, $\tilde{g}$ is Ricci-flat if and only if g is Einstein with Einstein constant $m-1$.

The Einstein condition is also very restrictive on Sasakian manifolds. The following proposition in [5] explains why this is.

Proposition 3.3.1. *On a Sasakian manifold,*

$$R\,(u, v)\,\mathcal{R} = \eta\,(v)\,u - \eta\,(u)\,v.$$

A Sasakian manifold is called Sasaki-Einstein if its Riemannian metric is Einstein. When computing the Ricci curvature, choose an orthonormal basis $\{v_j\}$ where $\eta\,(v_j) = 0$ together with $\mathcal{R}$. Then by plugging v_j's and $\mathcal{R}$ into the above proposition, we get

$$\mathrm{Ric}\,(v, \mathcal{R}) = 2n\eta\,(v)$$

if the Sasakian manifold is $2n+1$-dimensional. Therefore, if a Sasakian metric is Einstein, then its scalar curvature is positive and equal to $2n(n+1)$. Also, the cone metric of a Sasaki-Einstein manifold is Calabi-Yau in the sense that it is Ricci-flat. The immediate corollary of this is introduced in [7]:

Proposition 3.3.2. *Let Y be a complete Sasakian manifold such that* $\mathrm{Ric}\,(u, u) \geqslant \delta > -2$ *for all unit vector fields u on Y. Then Y is compact with finite fundamental group.*

The spaces of leaves of the characteristic foliation in Sasaki-Einstein manifolds satisfy some good properties. Although the quasi-regular Sasakian manifolds are as good as the regular ones regarding the space of leaves, let us restrict ourselves to the regular cases because it is more effective in understanding the ingredients. The following is the theorem in [7] about it.

Theorem 3.3.3. *Let (Y,g) be a compact regular Sasakian manifold of dimension $2n+1$, and let $\mathcal{Z}$ denote the space of leaves of the characteristic foliation. Then*

(i) $\mathcal{Z}$ is a compact complex manifold with a Kähler metric h and Kähler form ω which defines an integral class $[p^*\omega] \in H^2_{orb}(\mathcal{Z},\mathbb{Z})$ in such a way that $\pi : (Y,g) \longrightarrow (\mathcal{Z},h)$ is a Riemannian submersion. The fibers of π are totally geodesic submanifolds of Y diffeomorphic to S^1.

(ii) (Y,g) is Sasakian-Einstein if and only if $(\mathcal{Z},h)$ is Kähler-Einstein with scalar curvature $4n(n+1)$.

Before proving the theorem, let's take a look at some notions in Riemannian submersion $\pi : M \longrightarrow B$ introduced in [4]. For two vector fields E_1, E_2 in M, the (2,1) tensor fields T, A are defined by

$$T_{E_1}E_2 = \mathrm{hor}\left(\nabla_{\mathrm{ver}E_1}\left(\mathrm{ver}E_2\right)\right) + \mathrm{ver}\left(\nabla_{\mathrm{ver}E_1}\left(\mathrm{hor}E_2\right)\right),$$

$$A_{E_1}E_2 = \mathrm{hor}\left(\nabla_{\mathrm{hor}E_1}\left(\mathrm{ver}E_2\right)\right) + \mathrm{ver}\left(\nabla_{\mathrm{hor}E_1}\left(\mathrm{hor}E_2\right)\right).$$

Also, for an orthonormal basis $\{U_j\}$ of the vertical space, define

$$N = \sum_j T_{U_j}U_j.$$

Proof. (i) The fibers of π are totally geodesic because $\nabla_{\mathcal{R}}\mathcal{R} = 0$.

(ii) For two vectors u, v with $\eta(u) = \eta(v) = 0$,

$$A_u v = \mathrm{ver}\nabla_u v = g\left(\nabla_u v, \mathcal{R}\right) = -g\left(v, \nabla_u \mathcal{R}\right).$$

But by lemma 6.2 in [5], $\nabla_u\mathcal{R} = -\Phi u$, hence $A_u v = -g(\Phi u, v)\mathcal{R}$. Now, we are ready to use (9.36c) in [4]. Since the fibers of π are totally geodesic, $T \equiv 0$ (9.26 in [4]) and hence $T \equiv 0$. Therefore for an orthonormal frame $\{e_j\}$, $\eta(e_j) = 0$,

$$\mathrm{Ric}_g(u,v) = \mathrm{Ric}_h(u,v) - 2\sum_j g\left(A_u e_j, A_v e_j\right)$$

$$
\begin{aligned}
=&\mathrm{Ric}_h(u,v) - 2\sum_j g\left(g\left(\Phi u, e_j\right)\mathcal{R}, g\left(\Phi v, e_j\right)\mathcal{R}\right)\\
=&\mathrm{Ric}_h(u,v) - 2\sum_j g\left(\Phi u, e_j\right) g\left(\Phi v, e_j\right)\\
=&\mathrm{Ric}_h(u,v) - 2g\left(\Phi u, \Phi v\right)\\
=&\mathrm{Ric}_h(u,v) - 2g\left(u, v\right).
\end{aligned}
$$

Therefore, g is Einstein with Einstein constant $2n$ if and only if h is Einstein with Einstein constant $2(n+1)$.

□

This is true for quasi-regular Y except that $\mathcal{Z}$ becomes a compact complex orbifold.

Example 3.3.4. *On the Sasakian manifold $Y = S^{2n+1}$, its metric cone is $X = \mathbb{C}^{n+1}\backslash\{\mathbf{0}\}$, and the space of leaves is $\mathcal{Z} = \mathbb{C}P^n$ which are clearly Kähler. Here the submersion $\pi : Y \longrightarrow \mathcal{Z}$ is the well-known Hopf-fibration. Clearly,*

$$
R_{ij} = 2n\delta_{ij}
$$

on Y, and X is Ricci-flat. Also, the round metric induces the Fubini-Study metric g_{FS} on $\mathbb{C}P^n$ and

$$
\mathrm{Ric} = (2n+2)g_{FS}
$$

holds.

At this point, let's look into the metric cone $X = C(Y)$ of a Sasakian manifold (Y, g). As we know, X has a Kähler metric $\tilde{g} = dr^2 + r^2 g$. By direct computations, we see that its symplectic form ω is $\omega = -\frac{1}{2}d\left(r^2\eta\right)$. If we denote the complex structure on X by $\mathcal{I}$, clearly $\mathcal{R} = \mathcal{I}\left(r\frac{\partial}{\partial r}\right)$ and hence $\eta = \frac{dr}{r}\circ\mathcal{I} = d\log r \circ \mathcal{I}$ by taking their duals. Therefore,

$$
\eta\left(\frac{\partial}{\partial \bar{z}_j}\right) = d\log r\left(\mathcal{I}\frac{\partial}{\partial \bar{z}_j}\right) = d\log r\left(-i\frac{\partial}{\partial \bar{z}_j}\right) = -i\frac{\partial}{\partial \bar{z}_j}\left(\log r\right).
$$

Similarly,

$$
\eta\left(\frac{\partial}{\partial z_j}\right) = i\frac{\partial}{\partial z_j}\left(\log r\right).
$$

Therefore,

$$\eta = i\left(\partial - \bar{\partial}\right)\log r,$$

so that

$$d\eta = -2i\partial\bar{\partial}\log r.$$

Also,

$$\begin{aligned}
\omega =& -\frac{1}{2}\left(\partial + \bar{\partial}\right)\left(r^2 i\left(\partial - \bar{\partial}\right)\log r\right)\\
=& -\frac{1}{2}\left(\partial + \bar{\partial}\right)\left(r^2 i\left(\frac{\partial r - \bar{\partial} r}{r}\right)\right)\\
=& -\frac{i}{2}\left(\partial + \bar{\partial}\right)\left(r\partial r - r\bar{\partial} r\right)\\
=& -\frac{i}{2}\left(-2\partial r \wedge \bar{\partial} r - 2r\partial\bar{\partial} r\right)\\
=& i\partial\left(r\bar{\partial} r\right) = \frac{i}{2}\partial\bar{\partial} r^2.
\end{aligned}$$

We call $F = \frac{r^2}{4}$ a Kähler potential.

3.3.2 Toric Sasaki-Einstein manifolds

The main materials of this section are from [21]. Assume a Lie group G acts on a symplectic manifold M preserving its symplectic form. If this action is Hamiltonian, then a moment map for the action is a map $\mu : M \to \mathfrak{g}^*$ such that the formula $H_\xi(p) = \langle \mu(p), \xi \rangle$ defines a Lie algebra homomorphism $\xi \mapsto H_\xi$. To understand why this is called a moment map, let's take a look at an example in [23].

Example 3.3.5. *$G = SO(3)$ acts on $M = \mathbb{R}^3 \times \mathbb{R}^3$ by*

$$A : (x, y) \mapsto (Ax, Ay), \;\; A \in G.$$

If we identify $\mathbb{R}^3$ *with* $\mathfrak{g} = \mathfrak{so}(3)$ *by*

$$(x, y, z) \leftrightarrow \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix},$$

then the moment map $\mu : M \to \mathfrak{g}$ *is*

$$\mu(x, y) = A_{x \times y}.$$

$\vec{x} \times \vec{y}$ *is the angular momentum when* $\vec{x}$ *is the position and* $\vec{y}$ *is the force.*

We now impose (X, ω) is toric, where $X = C(Y)$, and Y is a $2n-1$-dimensional Sasakian manifold. This means the n-dimensional real torus $\mathbb{T}^n$ acts effectively on X, preserving the Kähler form. Actually, there is a theorem about the convexity of the image of moment maps of torus action on non-compact symplectic manifolds. For example, in [8]:

Theorem 3.3.6. *Let the torus* $\mathbb{T}^r$ *act in a Hamiltonian fashion on the symplectic manifold* (X, ω) *and denote* $\Phi : X \longrightarrow (\mathrm{Lie}\mathbb{T}^r)^* = \mathbb{R}^r$ *the corresponding moment map. Suppose that there exists a circle* $S^1 = \left\{e^{t\xi_0}\right\} \subset \mathbb{T}^r$ *for some* $\xi_0 \in \mathrm{Lie}\mathbb{T}^r$ *such that* $\Phi^{\xi_0} = \langle \Phi, \xi_0 \rangle$ *is a proper function having a minimum as its unique critical value. Then* $\Phi(X)$ *is the convex hull of a finite number of rays in* $(\mathrm{Lie}\mathbb{T}^r)^*$*.*

We require that the torus action is good enough to satisfy the conditions in the above theorem. Then its moment map $\mu : X \longrightarrow (\mathrm{Lie}\mathbb{T}^n)^* = \mathbb{R}^n$ is given by $\mu(\xi) = \left\langle -\frac{1}{2}r^2\eta, X_\xi \right\rangle$ because the symplectic form is $-\frac{1}{2}dr^2\eta$. This moment map allows to introduce symplectic coordinates y_i's on $\mathbb{R}^n$, where

$$y_i = \left\langle -\frac{1}{2}r^2\eta, \frac{\partial}{\partial \phi_i} \right\rangle,$$

and $\frac{\partial}{\partial \phi_i}$'s generate the $\mathbb{T}^n$ action. That is, ϕ_i are angular coordinates along the orbit of the torus action and let's say $\phi \sim \phi + 2\pi$.

Let $\mathcal{C} \subset \mathbb{R}^n$ be the image of the moment map. Because it is a strictly convex rational polyhedral cone, we may write

$$\mathcal{C} = \{y \in \mathbb{R}^n : l_a(y) = (y, v_a) \geqslant 0, a = 1, 2, \cdots, d\},$$

where v_a's are the inward normal vectors to the d facets of the polyhedral cone. Furthermore, we may assume v_a's are primitive elements of $\mathbb{Z}^n$.

To write the symplectic form ω in terms of y_j, ϕ_j, compute

$$\begin{aligned}
-2\omega\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial \phi_j}\right) =& d\left(r^2\eta\right)\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial \phi_j}\right) \\
=& \frac{\partial}{\partial y_i} r^2\eta\left(\frac{\partial}{\partial \phi_i}\right) - \frac{\partial}{\partial \phi_i} r^2\eta\left(\frac{\partial}{\partial y_i}\right) \\
=& -2\frac{\partial}{\partial y_i} y_j = \delta_{ij}.
\end{aligned}$$

Therefore $\omega = \sum_i dy_i \wedge d\phi_i$.

The metric is also $\mathbb{T}^n$-invariant, so we may write

$$ds^2 = \sum_{ij} G_{ij} dy_i \otimes dy_j + G^{ij} d\phi_i \otimes d\phi_j,$$

where $G_{ij}(y)$ are functions in y and G^{ij} is its inverse matrix. Then the complex structure $\mathcal{I}$ is

$$\mathcal{I} = \begin{bmatrix} 0 & -G^{ij} \\ G_{ij} & 0 \end{bmatrix}.$$

To check the integrability of $\mathcal{I}$,

$$\left[\mathcal{I}\frac{\partial}{\partial \phi_i}, \frac{\partial}{\partial \phi_j}\right] = -\sum_k \left[G^{ik}\frac{\partial}{\partial y_k}, \frac{\partial}{\partial \phi_j}\right] = 0,$$

because G^{ik} are independent of ϕ_j. Therefore we only need to check the vanishing of $\left[\mathcal{I}\frac{\partial}{\partial \phi_i}, \mathcal{I}\frac{\partial}{\partial \phi_j}\right]$.

$$\left[\mathcal{I}\frac{\partial}{\partial \phi_i}, \mathcal{I}\frac{\partial}{\partial \phi_j}\right] = \sum_{kl}\left[G^{il}\frac{\partial}{\partial y_l}, G^{jk}\frac{\partial}{\partial y_k}\right]$$

$$= \sum_{k,l} G^{il} G^{jk}_{,l} \frac{\partial}{\partial y_k} - G^{il}_{,k} G^{jk} \frac{\partial}{\partial y_l}$$
$$= \sum_{k,l} \left(G^{il} G^{jk}_{,l} - G^{jl} G^{ik}_{,l} \right) \frac{\partial}{\partial y_k}.$$

Consider
$$\sum_l G^{il} G^{jk}_{,l} = \sum_m G^{jm} G^{ik}_{,m}.$$

Multiply it by $G_{i\alpha}$ to get

$$\sum_{i,l} G^{il} G^{jk}_{,l} G_{i\alpha} = G^{jk}_{,\alpha} = \sum_{i,m} G^{jm} G^{ik}_{,m} G_{i\alpha} = -\sum_{i,m} G^{jm} G^{ik} G_{i\alpha,m}.$$

Multiply it by $G_{j\beta}$ to get

$$\begin{aligned}\sum_j G^{jk}_{,\alpha} G_{j\beta} &= -\sum_j G_{j\beta,\alpha} G^{jk} \\ &= -\sum_{m,i,j} G^{jm} G^{ik} G_{j\beta} G_{i\alpha,m} \\ &= -\sum_i G^{ik} G_{i\alpha,\beta}.\end{aligned}$$

So by multiplying it by $G_{k\alpha}$, we get $G_{\gamma\beta,\alpha} = G_{\gamma\alpha,\beta}$. Therefore

$$G_{ij} = \frac{\partial^2 G}{\partial y_i \partial y_j},$$

for some strictly convex function $G(y)$. We call it *the symplectic potential* for the Kähler metric. Actually, we have a canonical one defined by

$$G^{\text{can}}(y) = \frac{1}{2} \sum_a l_a(y) \log l_a(y).$$

Example 3.3.7. *On $\mathbb{C}^n$, $\mathbb{T}^n$ acts as an angular motion on each coordinate. In this action,*
$$y_j = \mu\left(\frac{\partial}{\partial \theta_j}\right) = -\frac{1}{2} r^2 \eta \left(\frac{\partial}{\partial \theta_j}\right) = \frac{r_j^2}{2},$$

Because $dy_j = r_j dr_j$ *and* $ds^2 = \sum_j dr_j^2 + r_j^2 d\theta_j^2$,

$$G_{ij} = \frac{1}{r_j^2}\delta_{ij} = \frac{2}{y_j}\delta_{ij},$$

$$G^{ij} = r_j^2\delta_{ij} = \frac{y_j}{2}\delta_{ij}.$$

And the cone is

$$\mathcal{C} = \{\mathbf{y} \in \mathbb{R}^n : y_j \geqslant 0, \forall j\}, \ v_a = e_a.$$

Now, let's consider the Reeb vector field $\mathcal{R}$. Then,

$$\begin{aligned}\mathcal{R}(y_i) =& \mathcal{L}_{\mathcal{R}}\left[-\frac{1}{2}r^2\eta\left(\frac{\partial}{\partial\phi_i}\right)\right] \\ =& \mathcal{L}_{\mathcal{R}}\left[-\frac{1}{2}r^2\eta\right]\left(\frac{\partial}{\partial\phi_i}\right) - \left(-\frac{1}{2}r^2\eta\right)\left(\mathcal{L}_{\mathcal{R}}\frac{\partial}{\partial\phi_i}\right) = 0.\end{aligned}$$

Hence we may write

$$\mathcal{R} = \sum_i b_i \frac{\partial}{\partial\phi_i}.$$

But

$$\begin{aligned}(dy_i)\left(r\frac{\partial}{\partial r}\right) =& r\frac{\partial}{\partial r}(y_i) \\ =& r\frac{\partial}{\partial r}\left(-\frac{1}{2}r^2\eta\left(\frac{\partial}{\partial\phi_i}\right)\right) \\ =& \mathcal{L}_{r\frac{\partial}{\partial r}}\left(-\frac{1}{2}r^2\eta\right)\left(\frac{\partial}{\partial\phi_i}\right) \\ =& -r^2\eta\left(\frac{\partial}{\partial\phi_i}\right) = 2y_i.\end{aligned}$$

Therefore $r\frac{\partial}{\partial r} = \sum_i 2y_i\frac{\partial}{\partial y_i}$. If we take $\mathcal{I}$ on the both sides, we get

$$b_i = \sum_j 2G_{ij}y_j.$$

It is straightforward to show that $b = (b_1, \cdots, b_n)$ is a constant vector. If we denote by b^{can} for the canonical symplectic potential, then

$$
\begin{aligned}
b_i^{\mathrm{can}} &= \sum_j G_{ij}^{\mathrm{can}} y_j = \sum_{j,a} \frac{v_i^a v_j^a}{l_a} y_j \\
&= \sum_a \frac{v_i^a}{l_a} \sum_j v_j^a y_j = \sum_a \frac{v_i^a}{l_a} l_a = \sum_a v_i^a .
\end{aligned}
$$

Now suppose the two symplectic potentials G, G' have the same Reeb vector, then $g = G - G'$ is

$$
g = \sum_i \lambda_i y_i - t,
$$

for some constants λ_i, t. Also, if we define

$$
G_b\left(y\right) = \frac{1}{2} l_b\left(y\right) \log l_b\left(y\right) - \frac{1}{2} l_\infty\left(y\right) \log l_\infty\left(y\right),
$$

where $l_\infty\left(y\right) = \left(b^{\mathrm{can}}, y\right)$, then the Reeb vector of G_b is $b - b^{\mathrm{can}}$. Therefore, we may summarize as follows:

Proposition 3.3.8. *The moduli space $\mathcal{S}$ of symplectic potentials corresponding to smooth Sasakian metrics on some fixed toric Sasakian manifold Y can be naturally written as*

$$
\mathcal{S} = \mathcal{C}_0^* \times \mathcal{H}(1)
$$

where $b \in \mathcal{C}_0^* \subset \mathbb{R}^n$ labels the Reeb vector for the Sasakian structure, and $g \in \mathcal{H}(1)$ is a smooth homogeneous degree one function on $\mathcal{C}$, such that G is strictly convex.

So far in dealing with the torus actions, we have not require and Calabi-Yau condition on X. Now we suppose the normal vectors to be

$$
v_a = (1, w_a)
$$

up to $SL(n; \mathbb{Z})$, which implies $c_1(X) = 0$. The Reeb vector has norm one at $r = 1$ reads

$$
1 = \sum_{i,j} b_i b_j G^{i,j} = \sum_{i,j,k} 2 b_i G_{jk} y_k G^{ij} = 2\left(b, y\right).
$$

Let

$$H = \left\{ y \in \mathbb{R}^n : (b, y) = \frac{1}{2} \right\} \cap \mathcal{C}.$$

to form a finite polytope $\Delta = \Delta_b$ which depends on b. Denote

$$X_1 = X|_{r \leqslant 1},$$

so that X_1 is a finite cone over the base Y. Because $r = 2\sum_j y_j$, we see that

$$\mu\left(X_1\right) = \Delta_b.$$

The volume is easily computed:

$$\operatorname{vol}\left(X_1\right) = \int_0^1 r^{2n-1} \operatorname{vol}\left(Y\right) dr = \frac{1}{2n} \operatorname{vol}\left(Y\right).$$

On the other hand,

$$\int_{\mu^{-1}(\Delta_b)} \frac{1}{n!} \omega^n = \int_{\Delta_b} dy_1 \cdots dy_n d\phi_1 \cdots d\phi_n = (2\pi)^n \operatorname{vol}\left(\Delta_b\right).$$

Therefore we get $\operatorname{vol}\left(Y\right) = 2n\left(2\pi\right)^n \operatorname{vol}\left(\Delta_b\right)$.

Example 3.3.9. *The area of unit sphere*

$$\begin{aligned} \operatorname{area}\left(S^{2n-1}\right) =& 2n\left(2\pi\right)^n \operatorname{vol}\left(\Delta_n\left(1/2\right)\right) \\ =& 2n\left(2\pi\right)^n \frac{1}{n!}\left(1/2\right)^n \\ =& \frac{(2\pi)^n}{2 \cdot 4 \cdots (2n-2)}. \end{aligned}$$

See the Figure 3.1.

Similarly,

$$\operatorname{vol}\left(\Sigma_a\right) = 2\left(n-1\right)\left(2\pi\right)^{n-1} \frac{1}{|v_a|} \operatorname{vol}\left(\mathcal{F}_a\right),$$

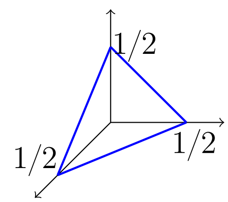

Figure 3.1: $\Delta_n(1/2)$

where $\mathcal{F}_a = \{y : (v_a, y) = 0\} \cap \{r \leqslant 1\}$, and $\Sigma_a = \mu^{-1}(\mathcal{F}_a)$. Also,

$$\int_\Delta R_X dy_1 \cdots dy_n = \sum_a \frac{2}{|v_a|} \mathrm{vol}(\mathcal{F}_a) - \frac{2n}{|b|} \mathrm{vol}(H),$$

so,

$$\int_{X_1} R_X = (2\pi)^n \int_\Delta R_X dy_1 \cdots dy_n = \frac{2\pi}{n-1} \sum_a \mathrm{vol}(\Sigma_a) - 2n\mathrm{vol}(Y).$$

For a Ricci-Flat Kähler manifold, set $R_X = 0$, then we get

$$2\pi \sum_a \mathrm{vol}(\Sigma_a) = n(n-1)\mathrm{vol}(Y).$$

Now, we begin with the Hilbert-Einstein action $S : Met(Y^{2n-1}) \to \mathbb{R}$ for a metric h on Y, where

$$S[h] = \int_Y (R_Y - 2(n-1)(2n-3))\, d\mu_Y,$$

and R_Y is the Ricci scalar of h. Then we already know that h is a critical point if and only if $\mathrm{Ric}_Y(h) = 2(n-1)h$, which is equivalent to the metric cone is Ricci-flat. Because

$$R_X = \frac{1}{r^2}[R_Y - (2n-1)(2n-2)],$$

if we integrate this over X_1, we get

$$\int_{X_1} R_X = \frac{1}{2n-2}\int_Y R_Y - (2n-1)(2n-2)d\mu.$$

Remarkably we see that the action depends only on b. Therefore we may interpret $S : \mathcal{C}^* \longrightarrow \mathbb{R}$, a function of b defined by

$$S[b] = 4\pi \sum_a \mathrm{vol}(\Sigma_a) - 4(n-1)^2 \mathrm{vol}(Y)$$

and the critical condition is

$$\frac{\partial}{\partial b_i} S = 0.$$

We can show that S is strictly convex on $\mathcal{C}_0^*$ hence has a unique critical point. Also, it is continuous and bounded below. Therefore, it has a unique minimum. While doing that, we need to prove $\mathrm{vol}(\Delta)$ is strictly convex on $\mathcal{C}_0^*$. Set $V(b) = \mathrm{vol}(\Delta)$. Then,

$$V = \int_\Delta dy_1 \cdots dy_n = \int_{\mathcal{C}} \theta\left(1 - 2(b,y)\right) dy_1 \cdots dy_n,$$

where

$$\theta(s) = \begin{cases} 0 & s \leqslant 0 \\ 1 & s > 0 \end{cases}.$$

Parametrize $y = th$, where $h \in H$. Then

$$ds = dy_1 \cdots dy_n = \frac{(b,h)}{\|b\|} t^{n-1} dt d\sigma.$$

So,

$$\begin{aligned}
\frac{\partial V}{\partial b_i} &= \int_{\mathcal{C}} \theta'\left(1 - 2(b,y)\right)(-2y_i)\, dy_1 \cdots dy_n \\
&= -2 \int_H \int_0^\infty \theta'\left(1 - 2(b,y)\right)(th_i) \frac{(b,h)}{\|b\|} t^{n-1} dt d\sigma \\
&= \int_H -2h_i \left(\frac{1}{2(b,h)}\right)^n \frac{(b,h)}{\|b\|} d\sigma
\end{aligned}$$

$$= -\int_H y_i \frac{1}{\|b\|} d\sigma$$

Note that $1 - 2(b, y) = 0$ if and only if $t = \frac{1}{2(b,y)} = 1$. Using this

$$\frac{\partial^2 V}{\partial b_i \partial b_j} = \frac{2(n+1)}{\|b\|} \int_H y_i y_j d\sigma,$$

which means V is strictly convex on $\mathcal{C}_0^*$.

Finally when $n = 3$,

$$\mathrm{vol}(Y) = \frac{\pi^3}{b_1} \sum_a \frac{(v_{a-1}, v_a, v_{a+1})}{(b, v_{a-1}, v_a)(b, v_a, v_{a+1})},$$

where (u, v, w) is the determinant of the 3×3 matrix whose rows are u, v, w.

3.3.3 Sasaki-Einstein metrics on $Y^{p,q}$

Concrete examples of irregular Sasaki-Einstein metrics are rarely known until [13] was published. They constructed toric Sasaki-Einstein metrics on $S^2 \times S^3$ in a very explicit but rather unfamiliar way to mathematicians. According to them, the AdS/CFT correspondence is one of the most important advancements in string theory because it provides a detailed correspondence between certain conformal field theories and geometries. A large class of examples consists of type IIB string theory on the background $AdS5L$, where L is a Sasaki–Einstein five–manifold and the dual theory is a four–dimensional $N = 1$ superconformal field theory. And they explain this is why the string theorists are so interested in Sasaki-Einstein geometry.

They start with giving the metric

$$ds^2 = \frac{1-cy}{6}\left(d\theta^2 + \sin^2\theta d\phi^2\right) + \frac{dy^2}{w(y)\,q(y)} + \frac{q(y)}{9}\left[d\psi - \cos\theta d\phi\right]^2$$

$$+w(y)\left[d\alpha + \frac{ac - 2y + y^2 c}{6(a - y^2)}(d\psi - \cos\theta d\phi)\right]^2$$

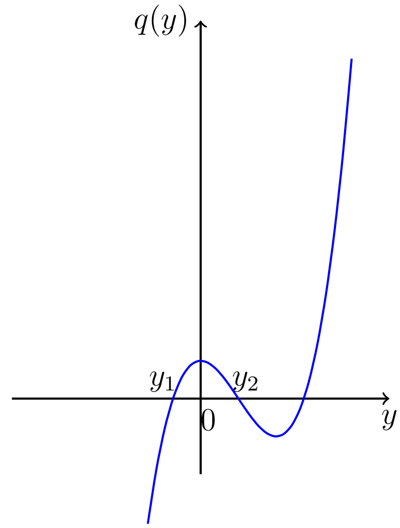

Figure 3.2: graph for $q(y)$

with

$$w(y) = \frac{2}{1-cy}$$
$$q(y) = \frac{a-3y^2+2cy^3}{a-y^2},$$

where $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi \leqslant 2\pi$. Also, they require $0 < a < 1$, and $y_1 \leqslant y \leqslant y_2$, where y_1 is the smallest zero of q and y_2 is the second one. And then they verify why the metrics fit for $S^2 \times S^3$. Let us follow their reasoning step by step. First set $c = 1$ for a while and see the first two term $\frac{1-y}{6}\left(d\theta^2 + \sin^2\theta d\phi^2\right)$. Clearly it is the usual round metric of S^2. Now fix θ and ϕ and see the following two terms $\frac{dy^2}{w(y)q(y)} + \frac{q(y)}{9}d\psi^2$. Because $y_1 \leqslant y \leqslant y_2$, q is non-negative. Easily see that $y_1 < 0$ and $y_2 > 0$ and $y_2 < \sqrt{a} < 1$. (See the figure 3.2) Therefore this metric on S^2 can be obtained from a multiple of the following parametrization:

$$(y, \psi) \mapsto \left(\sqrt{q(y)}\cos\psi, y, \sqrt{q(y)}\sin\psi\right),$$

where $y_1 \leqslant y \leqslant y_2$, $0 \leqslant \psi \leqslant 2\pi$.

Now, forget y for a moment and look at $\left(d\theta^2 + \sin^2\theta d\phi^2\right) + [d\psi - \cos\theta d\phi]^2$.

We want to regard $A = d\psi - \cos\theta d\phi$ as a connection form of some circle bundle over S^2. But because $d\phi$ is meaningless when $\theta = 0$ and $\theta = 2\pi$, we want it to vanish there. So, consider $A_+ = d\psi_+ - (\cos\theta - 1)\, d\phi$ for $\theta \leqslant \pi$, and $A_- = d\psi_- - (\cos\theta + 1)\, d\phi$ for $\theta \geqslant \pi$. On $\theta = \pi/2$, $A_+ = A_-$ implies that $d\psi_- = d\psi_+ + 2d\phi$, or $\psi_- = \psi_+ + 2\phi$. Therefore the winding number is 2 and it is a well-defined metric on the S^1-bundle over S^2 originated from the connection. Now if we count y together, then it is an S^2-bundle but the space of all S^2-bundles over S^2 is isomorphic to $\pi_1(SO(3)) = \mathbb{Z}_2$, thus it is a trivial bundle.

Now because we want to use the similar argument in order to build an S^1-bundle over $S^2 \times S^2$, let's write the metric as

$$ds^2 = ds^2(B_4) + w(y)(d\alpha + A)^2,$$

where $ds^2(B_4)$ is the metric on $S^2 \times S^2$ just described and

$$A = \frac{a - 2y + y^2}{6(a - y^2)}.$$

Let's say α runs from 0 to $2\pi l$ for some l and decide if there are appropriate l's to make S^1-bundles. Put

$$P_1 = \frac{1}{2\pi}\int_{\frac{S_1 - S_2}{2}} dA,\ P_2 = \frac{1}{2\pi}\int_{\frac{S_1 + S_2}{2}} dA,$$

where S_1 is the sphere of $S^2 \times \{pt\}$ and S_2 is $\{pt\} \times S^2$. Since the corresponding P_i/l must be an element in $H^2(S^2 \times S^2; \mathbb{Z})$, let's let

$$p = P_1/l,\ q = P_q/l,$$

and denote by $Y^{p,q}$ the total space of the S^1-bundle over $S^2 \times S^2$. Now what is left is to decide if there are appropriate a's to allow such l to exist. By simple computations, we see that

$$P_1 = \frac{y_1 - y_2}{6y_1y_2},\ P_2 = -\frac{(y_1 - y_2)^2}{9y_1y_2},$$

and hence our requirement is that $\lambda = y_2 - y_1 \in \mathbb{Q}$. Also,

$$y_1 = \frac{1}{2}\left(1 - \lambda - \sqrt{1 - \lambda^2/3}\right),$$

and

$$a = 3y_1^2(\lambda) - 2y_1^3(\lambda).$$

To summarize any rational value $\lambda = \frac{3q}{2p}$ is allowed within the range $0 < \lambda < \frac{3}{2}$ to get a such that $0 < a < 1$. l is also computed as

$$l = \frac{q}{3q^2 - 2p^2 + p\sqrt{4p^2 - 3q^2}}.$$

By using typical methods in algebraic topology, they show that $Y^{p,q}$ is always homeomorphic to $S^2 \times S^3$ for any positive integers p, q.

In order to see the Sasakian structure, let's let

$$\alpha = -\frac{1}{6}\beta - \frac{1}{6}\psi', \ \psi = \psi'.$$

Then the metric is rewritten as

$$ds^2 = \frac{1 - cy}{6}\left(d\theta^2 + \sin^2\theta d\phi^2\right) + \frac{dy^2}{w(y)\,q(y)} + \frac{w(y))q(y)}{36}\left[d\beta + c\cos\theta d\phi\right]^2$$

$$+\frac{1}{9}\left[d\psi' - \cos\theta + y\left(d\beta + c\cos\theta d\phi\right)\right]^2.$$

This is the standard form of

$$ds^2 = ds_4^4 + \left(\frac{1}{3}d\psi' + \sigma\right)^2,$$

and the Reeb vector field is

$$\mathcal{R} = \frac{\partial}{\partial\psi} - \frac{1}{6}\frac{\partial}{\partial\alpha},$$

and the contact form is

$$\eta = -6y\left[d\alpha + \frac{ac - 2y + y^2 c}{6\left(a - y^2\right)}\left(d\psi - \cos\theta d\phi\right)\right]$$

$$+\frac{a - 3y^2 + 2cy^3}{a - y^2}\left(d\psi - \cos\theta d\phi\right).$$

Therefore, the Sasakian structure in $Y^{p,q}$ is irregular unless l is rational.

In [20], they show that the moment map is

$$\mu = \left(\frac{r^2}{6}\left(1 - y\right)\left(\cos\theta - 1\right), \frac{r^2}{6}\left(1 - y\right)\cos\theta - \frac{r^2}{2}\left(p - q\right)ly, lr^2 y\right),$$

and the outward primitive normals are:

$$\begin{aligned} v_1 &= \left[1, 0, 0\right], \\ v_2 &= \left[1, -2, -\left(p + q\right)\right], \\ v_3 &= \left[1, -1, -p\right], \\ v_4 &= \left[1, -1, 0\right]. \end{aligned}$$

By using the method in the previous section, we get its volume:

$$\mathrm{vol}\left(Y^{p,q}\right) = \frac{q^2\left(2p + \sqrt{4p^2 - 3q^2}\right)}{3p^2\left(3q^2 - 2p^2 + p\sqrt{4p^2 - 3q^2}\right)}\pi^3.$$

For example, when $p = 2, q = 1$,

$$l = \frac{2\sqrt{13} + 5}{27},$$

so it is an irregular Sasaki-Einstein manifold. Also, its toric diagram is the figure 3.4. It happens to coincide to the diagram of the affine cone over the first del Pezzo surface $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$. Although it is known that $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ admits no Kähler-Einstein metric, still it admits a Sasakian-Einstein metric which is irregular.

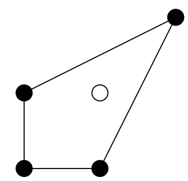

Figure 3.3: toric diagram for the Kähler cone over $Y^{2,1}$

3.3.4 Simple obstructions

In this section two simple but useful obstructions to the existence of Sasaki-Einstein metrics on the links of singularities will be overviewed that are introduced in [14]. To do that, we need some preliminaries. The first one is the Bishop inequality.

Theorem 3.3.10. *Let (M^n, g) be a compact Einstein manifold of positive Einstein constant normalized so that the metric cone $C(Y)$ is Ricci-flat, so* $\mathrm{Ric}(g) = (n-1)\,g$. *Then*

$$\mathrm{vol}\,(M^n, g) \leqslant \mathrm{vol}\,(S^n, g_{can}),$$

where g_{can} is the unit normal sphere metric on S^n.

Note that we've seen that the volume of a Sasakian metric on the link depends only on the Reeb vector field. Hence, we have the Bishop obstruction:

Theorem 3.3.11. *Let (X, Ω) be an isolated Gorenstein singularity with link L and putative Reeb vector field $\mathcal{R}$. If*

$$V(\mathcal{R}) = \frac{\mathrm{vol}\,(L, g_L)}{\mathrm{vol}\,(S^{2n-1}, g_{can})} > 1,$$

then X admits no Ricci-flat Kähler cone metric with Reeb vector field $\mathcal{R}$. In particular L does not admit a Sasaki-Einstein metric with this Reeb vector field.

Example 3.3.12. *Let L be any quasi-smooth link given by a weighted homo-*

geneous polynomial with weight vector $\mathbf{w}$ *and degree* d. *Then*

$$\operatorname{vol}(L) = \frac{2d}{\|\mathbf{w}\|(n-1)!}\left(\frac{\pi(|\mathbf{w}|-d)}{n}\right)^n,$$

hence

$$V(\mathcal{R}) = \frac{d(|\mathbf{w}|-d)^n}{n^n\|\mathbf{w}\|}.$$

Therefore, if

$$d(|\mathbf{w}|-d)^n > n^n\|\mathbf{w}\|,$$

then L *cannot admit any Sasaki-Einstein structure with this Reeb vector.*

The second obstruction needs the following theorem by Lichnerowicz.

Theorem 3.3.13. *Let* (M^n, g) *be a compact Riemannian manifold with* $\operatorname{Ric}(g) \geqslant n-1$. *Then the first non-zero eigenvalue* E_1 *of the Laplacian satisfies* $E_1 \geqslant n$. *Furthermore, the equality* $E_1 = n$ *happens only for manifolds isometric to the sphere.*

Therefore,

Theorem 3.3.14. *Let* (Y^{2n-1}, g) *be a compact manifold with s Sasaki-Einstein structure. Then the first non-zero eigenvalue* E_1 *of the Laplacian operator* Δ_g *is bounded*

$$E_1 \geqslant 2n-1$$

and $E_1 = 2n-1$ *if and only if it is the standard Sasaki-Einstein structure on* S^{2n-1}.

Let f be a holomorphic function on X with

$$\mathcal{L}_{\mathcal{R}} f = \lambda\sqrt{-1}f,$$

where $\lambda > 0$. Since f is holomorphic, $f = r^\lambda \tilde{f}$, where $\tilde{f}$ is the pull-back to X of a function L. Since X is Kähler, $\nabla_X^2 f = 0$, where

$$\nabla_X^2 = \frac{1}{r^2}\nabla_L^2 + \frac{1}{r^{2n-1}}\frac{1}{\partial r}\left(r^{2n-1}\frac{1}{\partial r}\right),$$

at $r = 1$. Note that

$$\frac{1}{r^{2n-1}}\frac{1}{\partial r}\left(r^{2n-1}\frac{1}{\partial r}\right) = (2n-1)\frac{1}{\partial r} + \frac{\partial^2}{\partial r^2}$$

at $r = 1$. Therefore if $\nabla_X^2 f = 0$, at $r = 1$

$$\begin{aligned} -\frac{1}{r^2}\nabla_L^2\left(r^\lambda \tilde{f}\right) &= -\nabla_L^2\left(\tilde{f}\right) \\ &= \left[(2n-1)\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}\right]\left(r^\lambda \tilde{f}\right) \\ &= (2n-1)\lambda\tilde{f} + \lambda(\lambda-1)\tilde{f} \end{aligned}$$

because $\frac{\partial \tilde{f}}{\partial r} = 0$. Consequently,

$$-\nabla_L^2\left(\tilde{f}\right) = E\tilde{f},$$

where $E = \lambda(\lambda + (2n-2))$. Clearly, $E \geqslant 2n-1$ if and only if $\lambda \geqslant 1$. Therefore,

Theorem 3.3.15. *Let (X, Ω) be an isolated Gorenstein singularity with link L and putative Reeb vector field $\mathcal{R}$. Suppose that there exists a holomorphic function f on X of positive $\lambda < 1$ under $\mathcal{R}$. Then X admits no Ricci-flat Kähler cone metric with Reeb vector field $\mathcal{R}$. In particular L does not admit a Sasaki-Einstein metric with this Reeb vector field.*

Let L be any quasi-smooth link given by a weighted homogeneous polynomial with weight vector $\mathbf{w}$ and degree d. From $f = r^\lambda \tilde{f}$, the holomorphic function with the smallest λ is z_m, where

$$w_m = \min\{w_i, i = 1, \cdots, n+1\},$$

and

$$\lambda = \frac{nw_m}{|\mathbf{w}| - d}.$$

Therefore, if

$$|\mathbf{w}| - d > nw_m,$$

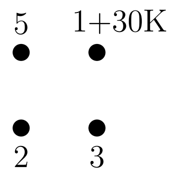

Figure 3.4: Brieskorn graph for $L\,(2,3,5,1+30k)$

then L cannot admit any Sasaki-Einstein structure with this Reeb vector.

Example 3.3.16. *Consider the links* $L\,(2,3,5,1+30k)$ *of Brieskorn polynomials with exponents* $(2,3,5,1+30k)$, $k=1,2,3,\cdots$. *These are homeomorphic to* S^5 *because their Brieskorn graphs have at least 2 isolated points (Refer to [18]). In fact, they are diffeomorphic to* S^5 *since there is no exotic 5-sphere. The weights* $\mathbf{w}$ *are*

$$\begin{aligned}\mathbf{w} &= \left(\frac{30\cdot(1+30k)}{2},\frac{30\cdot(1+30k)}{3},\frac{30\cdot(1+30k)}{5},\frac{30\cdot(1+30k)}{1+30k}\right)\\ &= (15\cdot(1+30k),10\cdot(1+30k),6\cdot(1+30k),30)\,.\end{aligned}$$

Also,

$$\begin{aligned}|\mathbf{w}|-d =&15\cdot(1+30k)+10\cdot(1+30k)+6\cdot(1+30k)+30-30\cdot(1+30k)\\ =&(1+30k)+30\end{aligned}$$

and

$$nw_m = 3\cdot 30 = 90.$$

Therefore $L\,(2,3,5,1+30k)$ *cannot admit a Sasaki-Einstein metric for* $k\geqslant 2$.

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국문초록

박사학위 논문으로서 본 논문은 크게 두 개의 장으로 이루어져 있습니다. 제 2 장에서는 저자가 서울대학교 수리과학부에서 학위를 하는 동안 출판한 논문에 대해 소개하였습니다. 구체적으로 Reeb 벡터장을 사교공간상의 경로로 간주하고 그 Conley-Zehnder 지표와 몫공간으로서 생성된 기저 공간의 orbifold 천(Chern) 특성류 사이의 관계를 규명하였습니다. 이렇게 얻어진 관계를 우리에게 매우 익숙한 기본적인 예제들에 적용시켜 구체적인 값을 구하였습니다.

제 3 장은 저자가 학위기간 동안 주로 연구한 분야인 사사키-아인슈타인 기하(Sasaki-Einstein geometry)에 대한 조사 보고서입니다. 기본적인 정의, 정리부터 흥미로운 예제, 존재성에 대한 걸림돌 이론(obstruction theory) 등에 대해서 살펴보았습니다.

주요어휘 : 콘리-젠더 지표, 오비폴드 천 특성류, 브리스코른 다항식, 사사키-아인슈타인 다양체

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